Richardson's Extrapolation

Suppose $\forall h \neq 0$ we have a formula $N_1(h)$ that approximates an unknown value M

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots , \tag{7}$$

for some unknown constants $K_1, K_2, K_3, \ldots$. If $K_1 \neq 0$, then the truncation error is $O(h)$. For example,

$$f'(x) - \frac{f(x+h) - f(x)}{h} = -\frac{f''(x)}{2!}h - \frac{f'''(x)}{3!}h^2 - \frac{f^{(4)}(x)}{4!}h^3 - \cdots .$$

Goal

Find an easy way to produce formulas with a higher-order truncation error.

Replacing h in (7) by $h/2$, we have

$$M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots . \tag{8}$$

Subtracting (7) with twice (8), we get

$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \cdots, \tag{9}$$

where

$$N_2(h) = 2N_1\left(\frac{h}{2}\right) - N_1(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right],$$

which is an $O(h^2)$ approximation formula.
Replacing h in (9) by $h/2$, we get

$$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8}h^2 - \frac{3K_3}{32}h^3 - \cdots. \tag{10}$$

Subtracting (9) from 4 times (10) gives

$$3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) + \frac{3K_3}{8}h^3 + \cdots,$$

which implies that

$$M = \left[N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3}\right] + \frac{K_3}{8}h^3 + \cdots \equiv N_3(h) + \frac{K_3}{8}h^3 + \cdots.$$

Using induction, M can be approximated by

$$M = N_m(h) + O(h^m),$$

where

$$N_m(h) = N_{m-1}\left(\frac{h}{2}\right) + \frac{N_{m-1}(h/2) - N_{m-1}(h)}{2^{m-1} - 1}.$$

Centered difference formula. From the Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \cdots$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \cdots$$

we have

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f^{(5)}(x) + \cdots,$$

and, consequently,

$$
\begin{aligned}
f'(x_0) &= \frac{f(x_0+h)-f(x_0-h)}{2h} - \left[\frac{h^2}{3!}f'''(x_0) + \frac{h^4}{5!}f^{(5)}(x_0) + \cdots\right], \\
&\equiv N_1(h) - \left[\frac{h^2}{3!}f'''(x_0) + \frac{h^4}{5!}f^{(5)}(x_0) + \cdots\right]. \qquad (11)
\end{aligned}
$$

Replacing h in (11) by $h/2$ gives

$$
f'(x_0) = N_1\left(\frac{h}{2}\right) - \frac{h^2}{24}f'''(x_0) - \frac{h^4}{1920}f^{(5)}(x_0) - \cdots . \qquad (12)
$$

Subtracting (11) from 4 times (12) gives

$$
f'(x_0) = N_2(h) + \frac{h^4}{480}f^{(5)}(x_0) + \cdots ,
$$

where

$$
N_2(h) = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \frac{N_1(h/2) - N_1(h)}{3}.
$$

In general,

$$f'(x_0) = N_j(h) + O(h^{2j})$$

with

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Example

Suppose that $x_0 = 2.0$, $h = 0.2$ and $f(x) = xe^x$. Compute an approximated value of $f'(2.0) = 22.16716829679195$ to six decimal places.

Solution. By centered difference formula, we have

$$\begin{aligned} N_1(0.2) &= \frac{f(2.0+0.2) - f(2.0-0.2)}{2h} = 22.414160, \\ N_1(0.1) &= \frac{f(2.0+0.1) - f(2.0-0.1)}{h} = 22.228786. \end{aligned}$$

It implies that

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = 22.166995$$

which does not have six decimal digits. Adding $N_1(0.05) = 22.182564$, we get

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = 22.167157$$

and

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = 22.167168$$

which contains six decimal digits.

$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$
1: $N_1(h) = N(h)$			
2: $N_1(h/2) = N(h/2)$	3: $N_2(h)$		
4: $N_1(h/4) = N(h/4)$	5: $N_2(h/2)$	6: $N_3(h)$	
7: $N_1(h/8) = N(h/8)$	8: $N_2(h/4)$	9: $N_3(h/2)$	10: $N_4(h)$

Remark

In practice, we are often encountered with the situation where the order of the numerical method is unknown. That is, the error expansion is of the form

$$M - N(h) = K_1 h^{p_1} + K_2 h^{p_2} + K_3 h^{p_3} + \cdots , \tag{13}$$

where $p_1, p_2, \cdots$ are unknown. Solving for the leading order p_1, together with the primary unknowns M and K_1, requires 3 equations, which can be obtained from, for example, the numerical results at h, $h/2$ and $h/4$:

$$\begin{array}{rcl} M - N(h) & = & K_1 h^{p_1} + \cdots , \\ M - N(\frac{h}{2}) & = & K_1 \left(\frac{h}{2}\right)^{p_1} + \cdots , \\ M - N(\frac{h}{4}) & = & K_1 \left(\frac{h}{4}\right)^{p_1} + \cdots \end{array} \tag{14}$$

The answer is given by

$$p_1 \approx \log_2 \frac{N(h) - N(\frac{h}{2})}{N(\frac{h}{2}) - N(\frac{h}{4})}$$

Once p_1 is known, Richardson extrapolation can be proceeded as before.