Newton's method

Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f \in C^2[a, b]$, i.e., f'' exists and is continuous. If $f(x^*) = 0$ and $x^* = x + h$ where h is small, then by Taylor's theorem

$$\begin{aligned} 0 = f(x^*) &= f(x+h) \\ &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots \\ &= f(x) + f'(x)h + O(h^2). \end{aligned}$$

Since h is small, $O(h^2)$ is negligible. It is reasonable to drop $O(h^2)$ terms. This implies

$$f(x) + f'(x)h \approx 0 \quad \text{and} \quad h \approx -\frac{f(x)}{f'(x)}, \quad \text{if} \quad f'(x) \neq 0.$$

Hence

$$x + h = x - \frac{f(x)}{f'(x)}$$

is a better approximation to x^*.

This sets the stage for the Newton-Raphson's method, which starts with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since the Taylor's expansion of $f(x)$ at x_k is given by

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2} f''(x_k)(x - x_k)^2 + \cdots .$$

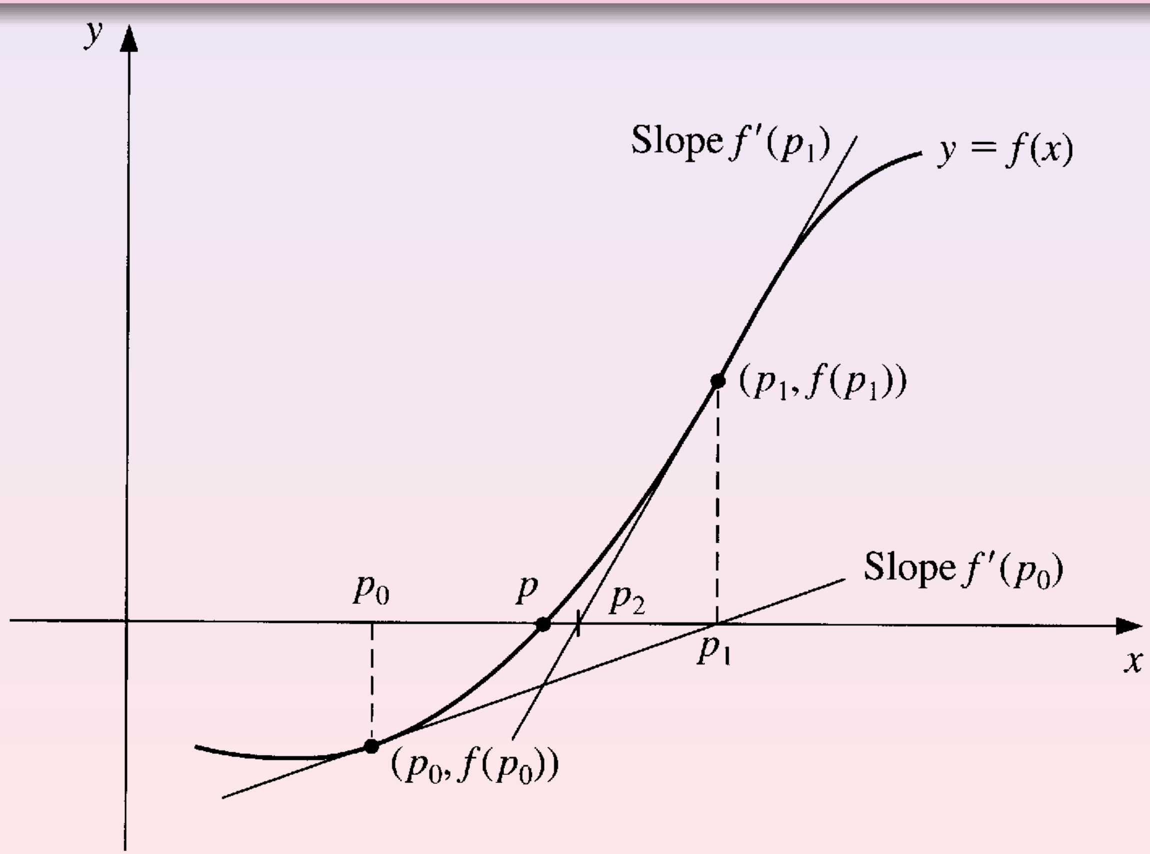
At x_k, one uses the tangent line

$$y = \ell(x) = f(x_k) + f'(x_k)(x - x_k)$$

to approximate the curve of $f(x)$ and uses the zero of the tangent line to approximate the zero of $f(x)$.

Newton's Method

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$.
While $i \leq M$ and $|x - x_0| \geq TOL$
 Set $i = i + 1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$.
End While



Three stopping-technique inequalities

$$
\begin{aligned}
(a). \quad & |x_n - x_{n-1}| < \varepsilon, \\
(b). \quad & \frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon, \quad x_n \neq 0, \\
(c). \quad & |f(x_n)| < \varepsilon.
\end{aligned}
$$

Note that Newton's method for solving $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for} \quad n \geq 1$$

is just a special case of functional iteration in which

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Example

The following table shows the convergence behavior of Newton's method applied to solving $f(x) = x^2 - 1 = 0$. Observe the quadratic convergence rate.

n	x_n	$\lvert e_n\rvert \equiv \lvert 1 - x_n\rvert$
0	2.0	1
1	1.25	0.25
2	1.025	2.5e-2
3	1.0003048780488	3.048780488e-4
4	1.0000000464611	4.64611e-8
5	1.0	0

Theorem

Assume $f(x^*) = 0$, $f'(x^*) \neq 0$ *and* $f(x)$, $f'(x)$ *and* $f''(x)$ *are continuous on* $N_\varepsilon(x^*)$. *Then if* x_0 *is chosen sufficiently close to* x^*, *then*

$$\left\{ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \right\} \to x^*.$$

Proof: Define

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Find an interval $[x^* - \delta, x^* + \delta]$ such that

$$g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$$

and

$$|g'(x)| \leq k < 1, \ \forall \ x \in (x^* - \delta, x^* + \delta).$$

Since f' is continuous and $f'(x^*) \neq 0$, it implies that $\exists\ \delta_1 > 0$ such that $f'(x) \neq 0\ \forall\ x \in [x^* - \delta_1, x^* + \delta_1] \subseteq [a, b]$. Thus, g is defined and continuous on $[x^* - \delta_1, x^* + \delta_1]$. Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [x^* - \delta_1, x^* + \delta_1]$. Since f'' is continuous on $[a, b]$, we have g' is continuous on $[x^* - \delta_1, x^* + \delta_1]$.
By assumption $f(x^*) = 0$, so

$$g'(x^*) = \frac{f(x^*)f''(x^*)}{|f'(x^*)|^2} = 0.$$

Since g' is continuous on $[x^* - \delta_1, x^* + \delta_1]$ and $g'(x^*) = 0$, $\exists\ \delta$ with $0 < \delta < \delta_1$ and $k \in (0, 1)$ such that

$$|g'(x)| \leq k,\ \forall\ x \in [x^* - \delta, x^* + \delta].$$

Claim: $g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$.
If $x \in [x^* - \delta, x^* + \delta]$, then, by the Mean Value Theorem, $\exists\ \xi$ between x and x^* such that

$$|g(x) - g(x^*)| = |g'(\xi)||x - x^*|.$$

It implies that

$$\begin{aligned} |g(x) - x^*| &= |g(x) - g(x^*)| = |g'(\xi)||x - x^*| \\ &\leq k|x - x^*| < |x - x^*| < \delta. \end{aligned}$$

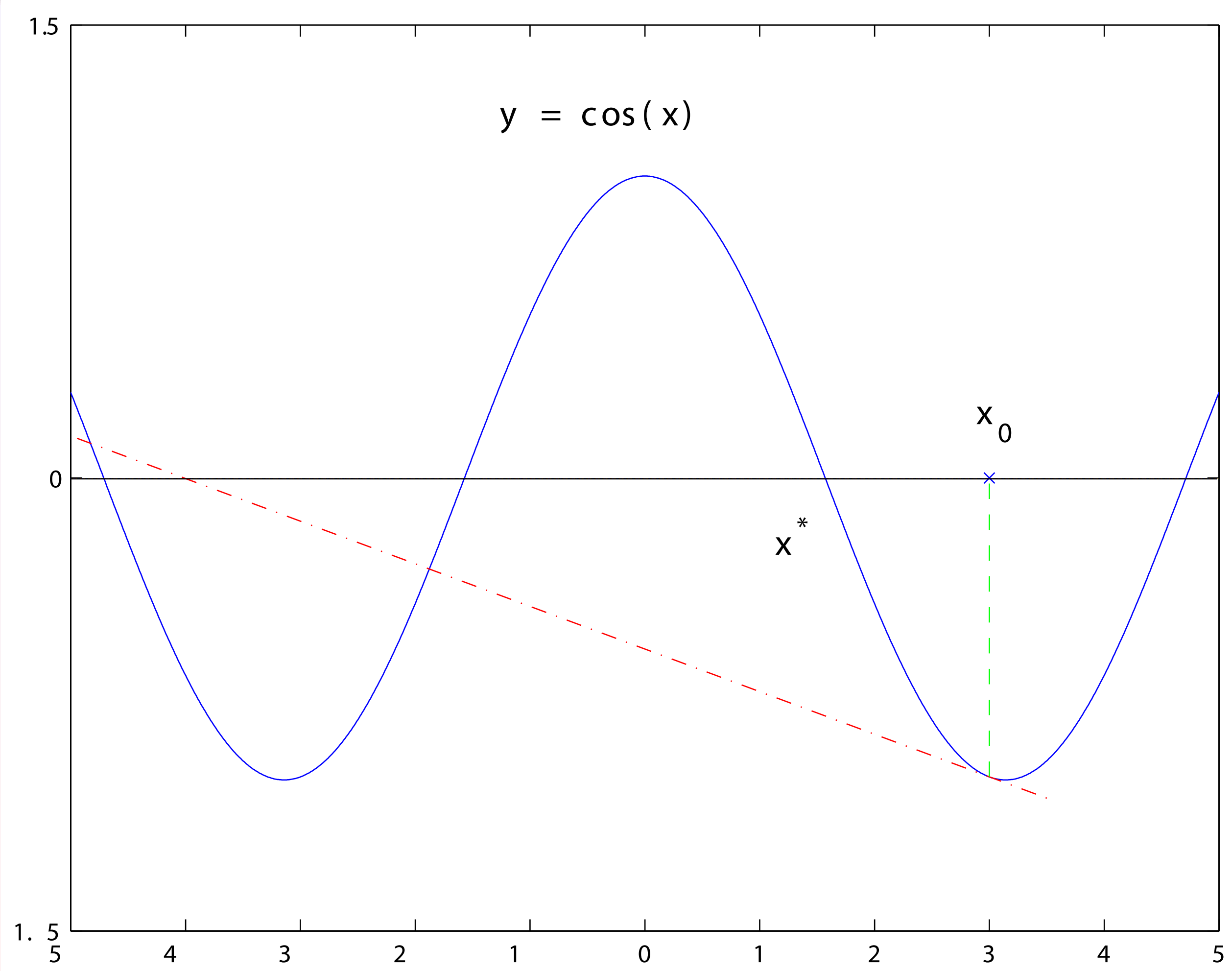
Hence, $g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta]$.
By the Fixed-Point Theorem, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = g(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad \text{for} \quad n \geq 1,$$

converges to x^* for any $x_0 \in [x^* - \delta, x^* + \delta]$.

Example

When Newton's method applied to $f(x) = \cos x$ with starting point $x_0 = 3$, which is close to the root $\frac{\pi}{2}$ of f, it produces $x_1 = -4.01525, x_2 = -4.8526, \cdots$, which converges to another root $-\frac{3\pi}{2}$.



Secant method

Disadvantage of Newton's method

In many applications, the derivative $f'(x)$ is very expensive to compute, or the function $f(x)$ is not given in an algebraic formula so that $f'(x)$ is not available.

By definition,

$$f'(x_{n-1}) = \lim_{x \longrightarrow x_{n-1}} \frac{f(x) - f(x_{n-1})}{x - x_{n-1}}.$$

Letting $x = x_{n-2}$, we have

$$f'(x_{n-1}) \approx \frac{f(x_{n-2}) - f(x_{n-1})}{x_{n-2} - x_{n-1}} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

Using this approximation for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})},$$

which is called the Secant method.

From geometric point of view, we use a secant line through x_{n-1} and x_{n-2} instead of the tangent line to approximate the function at the point x_{n-1}. The slope of the secant line is

$$s_{n-1} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$

and the equation is

$$M(x) = f(x_{n-1}) + s_{n-1}(x - x_{n-1}).$$

The zero of the secant line

$$x = x_{n-1} - \frac{f(x_{n-1})}{s_{n-1}} = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

is then used as a new approximate x_n.

Secant Method

Given x_0, x_1, tolerance TOL, maximum number of iteration M.

Set $i = 2$; $y_0 = f(x_0); y_1 = f(x_1)$;

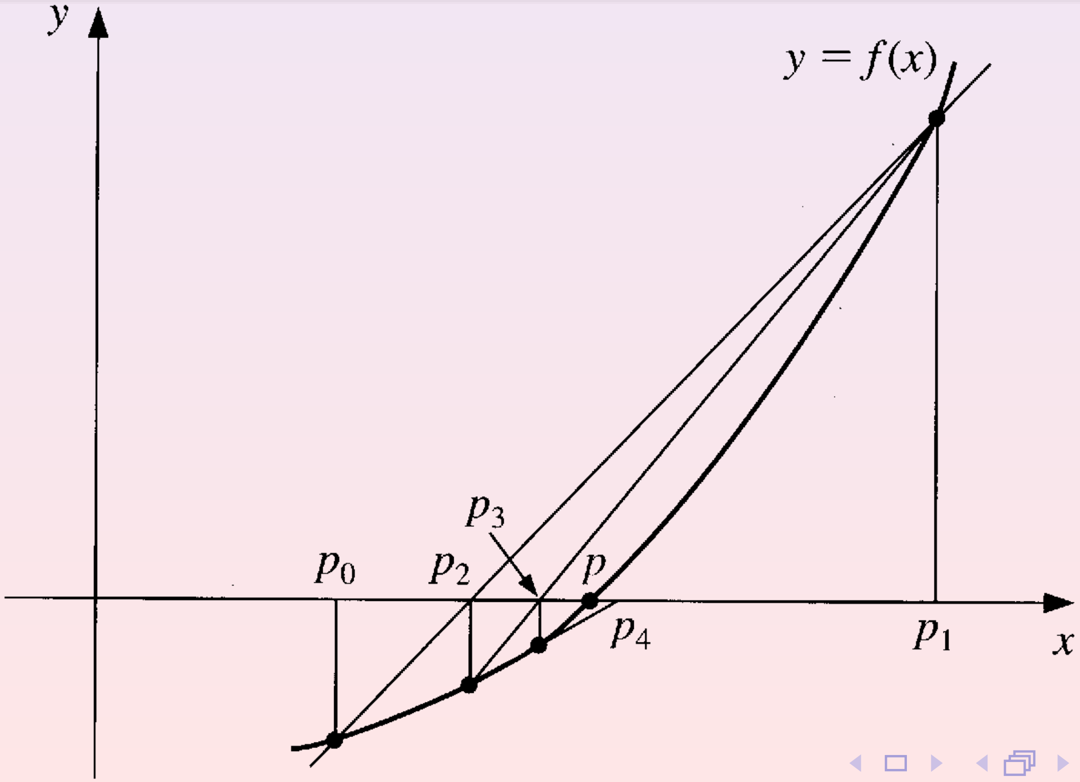
$x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.

While $i \leq M$ and $|x - x_1| \geq TOL$

Set $i = i + 1$; $x_0 = x_1; y_0 = y_1; x_1 = x; y_1 = f(x)$;

$x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.

End While



Method of False Position

1. Choose initial approximations x_0 and x_1 with $f(x_0)f(x_1) < 0$.
2. $x_2 = x_1 - f(x_1)(x_1 - x_0)/(f(x_1) - f(x_0))$
3. Decide which secant line to use to compute x_3:
   If $f(x_2)f(x_1) < 0$, then x_1 and x_2 bracket a root, i.e.,

   $$x_3 = x_2 - f(x_2)(x_2 - x_1)/(f(x_2) - f(x_1))$$

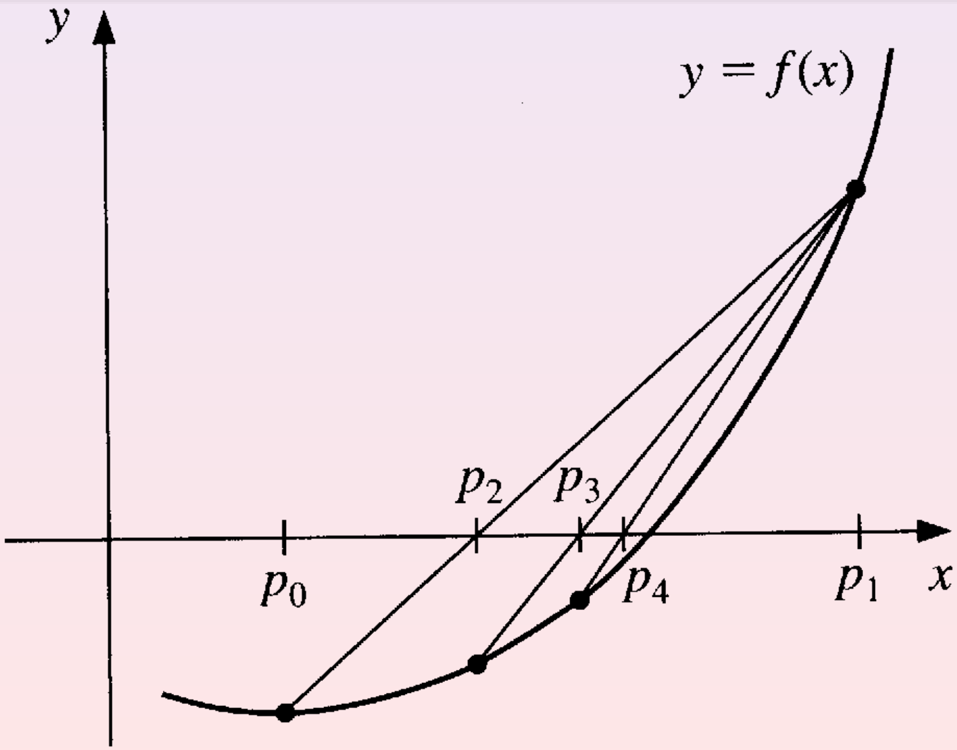
   Else, x_0 and x_2 bracket a root, i.e.,

   $$x_3 = x_2 - f(x_2)(x_2 - x_0)/(f(x_2) - f(x_0))$$

   End if

Method of False Position

Given x_0, x_1, tolerance TOL, maximum number of iteration M.
Set $i = 2$; $y_0 = f(x_0); y_1 = f(x_1)$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
While $i \leq M$ and $|x - x_1| \geq TOL$
    Set $i = i + 1$; $y = f(x)$.
    If $y \cdot y_1 < 0$, then set $x_0 = x_1; y_0 = y_1$.
    Set $x_1 = x; y_1 = y$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
End While



Error analysis for iterative methods

Definition

Let $\{x_n\} \to x^*$. If there are positive constants c and α such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = c,$$

then we say the rate of convergence is of order α.

We say that the rate of convergence is

1. linear if $\alpha = 1$ and $0 < c < 1$.
2. superlinear if
$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = 0;$$
3. quadratic if $\alpha = 2$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ and $\{\tilde{x}_n\}_{n=0}^{\infty}$ are linearly and quadratically convergent to x^*, respectively, with the same constant $c = 0.5$. For simplicity, suppose that

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} \approx c \quad \text{and} \quad \frac{|\tilde{x}_{n+1} - x^*|}{|\tilde{x}_n - x^*|^2} \approx c.$$

These imply that

$$|x_n - x^*| \approx c|x_{n-1} - x^*| \approx c^2|x_{n-2} - x^*| \approx \cdots \approx c^n|x_0 - x^*|,$$

and

$$\begin{aligned} |\tilde{x}_n - x^*| &\approx c|\tilde{x}_{n-1} - x^*|^2 \approx c\left[c|\tilde{x}_{n-2} - x^*|^2\right]^2 = c^3|\tilde{x}_{n-2} - x^*|^4 \\ &\approx c^3\left[c|\tilde{x}_{n-3} - x^*|^2\right]^4 = c^7|\tilde{x}_{n-3} - x^*|^8 \\ &\approx \cdots \approx c^{2^n-1}|\tilde{x}_0 - x^*|^{2^n}. \end{aligned}$$

Remark

Quadratically convergent sequences generally converge much more quickly than those that converge only linearly.

Theorem

Let $g \in C[a,b]$ *with* $g([a,b]) \subseteq [a,b]$. *Suppose that* g' *is continuous on* (a,b) *and* $\exists\ k \in (0,1)$ *such that*

$$|g'(x)| \leq k, \ \forall\ x \in (a,b).$$

If $g'(x^*) \neq 0$, *then for any* $x_0 \in [a,b]$, *the sequence*

$$x_n = g(x_{n-1}), \quad \textit{for} \quad n \geq 1$$

converges only linearly to the unique fixed point x^* *in* $[a,b]$.

Proof:

- By the Fixed-Point Theorem, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^*.
- Since g' exists on (a, b), by the Mean Value Theorem, $\exists\ \xi_n$ between x_n and x^* such that

$$x_{n+1} - x^* = g(x_n) - g(x^*) = g'(\xi_n)(x_n - x^*).$$

- $\because \{x_n\}_{n=0}^{\infty} \to x^* \quad \Rightarrow \quad \{\xi_n\}_{n=0}^{\infty} \to x^*$
- Since g' is continuous on (a, b), we have

$$\lim_{n\to\infty} g'(\xi_n) = g'(x^*).$$

- Thus,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \lim_{n\to\infty} |g'(\xi_n)| = |g'(x^*)|.$$

Hence, if $g'(x^*) \neq 0$, fixed-point iteration exhibits linear convergence.

Theorem

Let x^* be a fixed point of g and I be an open interval with $x^* \in I$. Suppose that $g'(x^*) = 0$ and g'' is continuous with

$$|g''(x)| < M, \ \forall \ x \in I.$$

Then $\exists \ \delta > 0$ such that

$$\{x_n = g(x_{n-1})\}_{n=1}^{\infty} \ \rightarrow \ x^* \quad \text{for} \quad x_0 \in [x^* - \delta, x^* + \delta]$$

at least quadratically. Moreover,

$$|x_{n+1} - x^*| < \frac{M}{2}|x_n - x^*|^2, \quad \text{for sufficiently large} \quad n.$$

Proof:

- Since $g'(x^*) = 0$ and g' is continuous on I, $\exists\ \delta$ such that $[x^* - \delta, x^* + \delta] \subset I$ and

$$|g'(x)| \leq k < 1, \ \forall\ x \in [x^* - \delta, x^* + \delta].$$

- In the proof of the convergence for Newton's method, we have

$$\{x_n\}_{n=0}^{\infty} \subset [x^* - \delta, x^* + \delta].$$

- Consider the Taylor expansion of $g(x_n)$ at x^*

$$\begin{aligned} x_{n+1} = g(x_n) &= g(x^*) + g'(x^*)(x_n - x^*) + \frac{g''(\xi)}{2}(x_n - x^*)^2 \\ &= x^* + \frac{g''(\xi)}{2}(x_n - x^*)^2, \end{aligned}$$

where ξ lies between x_n and x^*.

- Since

$$|g'(x)| \le k < 1, \ \forall\, x \in [x^* - \delta, x^* + \delta]$$

  and

$$g([x^* - \delta, x^* + \delta]) \subseteq [x^* - \delta, x^* + \delta],$$

  it follows that $\{x_n\}_{n=0}^{\infty}$ converges to x^*.
- But ξ_n is between x_n and x^* for each n, so $\{\xi_n\}_{n=0}^{\infty}$ also converges to x^* and

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^2} = \frac{|g''(x^*)|}{2} < \frac{M}{2}.$$

- It implies that $\{x_n\}_{n=0}^{\infty}$ is quadratically convergent to x^* if $g''(x^*) \neq 0$ and

$$|x_{n+1} - x^*| < \frac{M}{2}|x_n - x^*|^2, \quad \text{for sufficiently large} \quad n. \qquad \Box$$

Example

Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ corresponds to $g(x) = x - \frac{f(x)}{f'(x)}$. Suppose that $f(x)$ has a m-fold root at x^*, that is

$$f(x) = (x - x^*)^m q(x), \qquad q(x^*) \neq 0.$$

Let $\mu(x) = \frac{f(x)}{f'(x)} = (x - x^*)\frac{q(x)}{mq(x)+(x-x^*)q'(x)}$, it is easy to see that $\mu'(x^*) = \frac{1}{m}$. It follows that $0 \leq g'(x_*) = 1 - \frac{1}{m} < 1$. Hence Newton's method is locally convergent. Moreover, it converges quadratically for simple roots ($m = 1$) and linearly for multiple roots ($m > 1$).

Remedy for slow convergence on multiple roots ($m > 1$):

- If m is known, take $x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)}$.
- If m is not known, take $x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)}$, since $\mu(x) = \frac{f(x)}{f'(x)} = \frac{O(x-x^*)^m}{O(x-x^*)^{m-1}} = O(x - x^*)$ always has a simple root at x^* for any $m \geq 1$. This is known as modified Newton's method.

Global Convergence for Convex (Concave) Functions

Theorem

If $f \in C^2$, $f'' > 0$ and $f(x) = 0$ has a root, then Newton's method always converges to a root x^* for any initial x_0.

Proof:

It suffices to consider the case where $f' > 0$, $f'' > 0$ and $f(x) = 0$ has a root. In this case, the root x^* is unique. Define $e_n = x_n - x^*$. Since $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. It follows that

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)}. \qquad (2)$$

Moreover, since $f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(\xi_n)}{2}(x^* - x_n)^2$, we also have $f(x_n) = f'(x_n)e_n - \frac{f''(\xi_n)}{2}e_n^2$. Therefore

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} = \frac{f''(\xi_n)}{2f'(x_n)}e_n^2 > 0. \qquad (3)$$

Consequently $x_{n+1} > x^*$ and $f(x_{n+1}) > 0$ for all $n \geq 0$.

Moreover $e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} < e_n$, we conclude that

$$0 < ... < x_{n+1} < x_n < ... < x_1$$

and x_n converges monotonically to some $\tilde{x}$ satisfying $\tilde{x} = \tilde{x} - \frac{f(\tilde{x})}{f'(\tilde{x})}$, that is $f(\tilde{x}) = 0$, thus $\tilde{x} = x^*$ by uniqueness of the root.

The proof for other cases

- $f' < 0$, $f'' > 0$, $f(x) = 0$ has a root.
- $f'' > 0$, has two distinct roots.
- $f'' > 0$, has a double root. are similar. So is the concave case ($f'' < 0$).

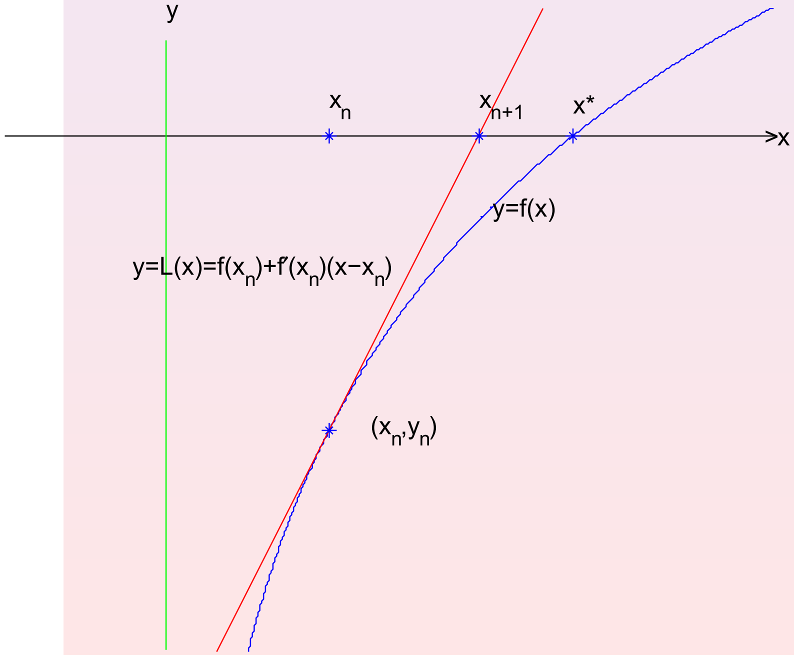
Alternative Error Estimate for Newton's Method

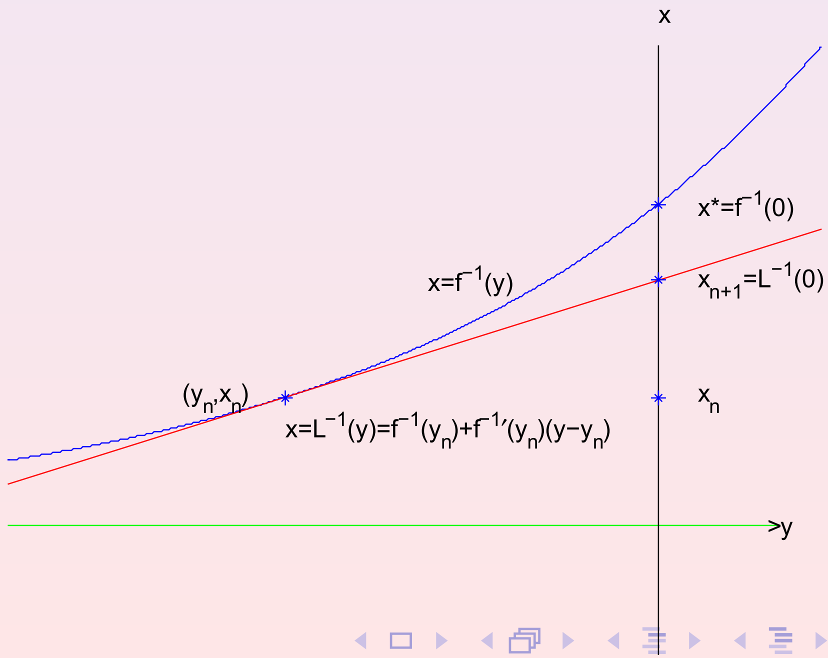
Suppose $f'(x^*) \neq 0$, then both $f(x)$ and its linearization at (x_n, y_n), $L(x)$, are locally invertible (Inverse Function Theorem). The formula of the tangent lines are given by

$$L(x) = f(x_n) + \frac{df(x_n)}{dx}(x - x_n)$$

and

$$L^{-1}(y) = f^{-1}(y_n) + \frac{df^{-1}(y_n)}{dy}(y - y_n) = x_n + \frac{1}{f'(x_n)}(y - y_n)$$





Since $x^* = f^{-1}(0)$ and $x_{n+1} = L^{-1}(0)$, the error estimate for Newton's method reduces to error estimate between $f^{-1}(y)$ and its linearization approximation $L^{-1}(y)$ at $y = 0$. From standard analysis, the error is proportional to $(0 - y_n)^2$:

$$|x_{n+1} - x^*| = |L^{-1}(0) - f^{-1}(0)| = \frac{1}{2}\left|\frac{d^2 f^{-1}}{dy^2}(\eta_n)(y_n - 0)^2\right|$$

$$= \frac{1}{2}\left|\frac{d^2 f^{-1}}{dy^2}(\eta_n)\right|\left(f(x_n) - f(x^*)\right)^2 = \left(\frac{1}{2}\left|\frac{d^2 f^{-1}}{dy^2}(\eta_n)\right| \cdot \left(f'(\xi_n)\right)^2\right)(x_n - x^*)^2$$

The main advantage of this formulation:
Higher order approximations of $f^{-1}(0)$, such as quadratic approximation, gives rise to higher order iteration schemes for solving the original equation $f(x) = 0$.

Error Analysis of Secant Method

Reference: D. Kincaid and W. Cheney, "Numerical analysis"
Let x^* denote the exact solution of $f(x) = 0$, $e_k = x_k - x^*$ be the errors at the k-th step. Then

$$\begin{aligned}
e_{k+1} &= x_{k+1} - x^* \\
&= x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} - x^* \\
&= \frac{1}{f(x_k) - f(x_{k-1})}\left[(x_{k-1} - x^*)f(x_k) - (x_k - x^*)f(x_{k-1})\right] \\
&= \frac{1}{f(x_k) - f(x_{k-1})}\left(e_{k-1}f(x_k) - e_k f(x_{k-1})\right) \\
&= e_k e_{k-1}\left(\frac{\frac{1}{e_k}f(x_k) - \frac{1}{e_{k-1}}f(x_{k-1})}{x_k - x_{k-1}} \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right)
\end{aligned}$$

To estimate the numerator $\frac{\frac{1}{e_k}f(x_k)-\frac{1}{e_{k-1}}f(x_k)}{x_k-x_{k-1}}$, we apply the Taylor's theorem

$$f(x_k) = f(x^* + e_k) = f(x^*) + f'(x^*)e_k + \frac{1}{2}f''(x^*)e_k^2 + O(e_k^3),$$

to get

$$\frac{1}{e_k}f(x_k) = f'(x^*) + \frac{1}{2}f''(x^*)e_k + O(e_k^2).$$

Similarly,

$$\frac{1}{e_{k-1}}f(x_{k-1}) = f'(x^*) + \frac{1}{2}f''(x^*)e_{k-1} + O(e_{k-1}^2).$$

Hence

$$\frac{1}{e_k}f(x_k) - \frac{1}{e_{k-1}}f(x_{k-1}) \approx \frac{1}{2}(e_k - e_{k-1})f''(x^*).$$

Since $x_k - x_{k-1} = e_k - e_{k-1}$ and

$$\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \to \frac{1}{f'(x^*)},$$

we have

$$
\begin{aligned}
e_{k+1} &\approx e_k e_{k-1} \left(\frac{\frac{1}{2}(e_k - e_{k-1}) f''(x^*)}{e_k - e_{k-1}} \cdot \frac{1}{f'(x^*)} \right) = \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} e_k e_{k-1} \\
&\equiv C e_k e_{k-1} \qquad (4)
\end{aligned}
$$

To estimate the convergence rate, we assume

$$
|e_{k+1}| \approx \eta |e_k|^\alpha,
$$

where $\eta > 0$ and $\alpha > 0$ are constants, i.e.,

$$
\frac{|e_{k+1}|}{\eta |e_k|^\alpha} \to 1 \quad \text{as} \quad k \to \infty.
$$

Then $|e_k| \approx \eta |e_{k-1}|^\alpha$ which implies $|e_{k-1}| \approx \eta^{-1/\alpha} |e_k|^{1/\alpha}$. Hence (4) gives

$$
\eta |e_k|^\alpha \approx C |e_k| \eta^{-1/\alpha} |e_k|^{1/\alpha} \quad \Longrightarrow \quad C^{-1} \eta^{1+\frac{1}{\alpha}} \approx |e_k|^{1-\alpha+\frac{1}{\alpha}}.
$$

Since $|e_k| \to 0$ as $k \to \infty$, and $C^{-1}\eta^{1+\frac{1}{\alpha}}$ is a nonzero constant,

$$
1 - \alpha + \frac{1}{\alpha} = 0 \quad \Longrightarrow \quad \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.
$$

This result implies that $C^{-1}\eta^{1+\frac{1}{\alpha}} \longrightarrow 1$ and

$$\eta \longrightarrow C^{\frac{\alpha}{1+\alpha}} = \left(\frac{f''(x^*)}{2f'(x^*)}\right)^{0.62}.$$

In summary, we have shown that

$$|e_{k+1}| = \eta |e_k|^{\alpha}, \quad \alpha \approx 1.62,$$

that is, the rate of convergence is superlinear.
Rate of convergence:

- secant method: superlinear
- Newton's method: quadratic
- bisection method: linear

Each iteration of method requires

- secant method: one function evaluation
- Newton's method: two function evaluation, namely, $f(x_k)$ and $f'(x_k)$.
  $\Rightarrow$ two steps of secant method are comparable to one step of Newton's method. Thus

$$|e_{k+2}| \approx \eta |e_{k+1}|^{\alpha} \approx \eta^{1+\alpha} |e_k|^{\frac{3+\sqrt{5}}{2}} \approx \eta^{1+\alpha} |e_k|^{2.62}.$$

  $\Rightarrow$ secant method is more efficient than Newton's method.

Remark

Two steps of secant method would require a little more work than one step of Newton's method.