Numerical Analysis I
Interpolation and Polynomial Approximation

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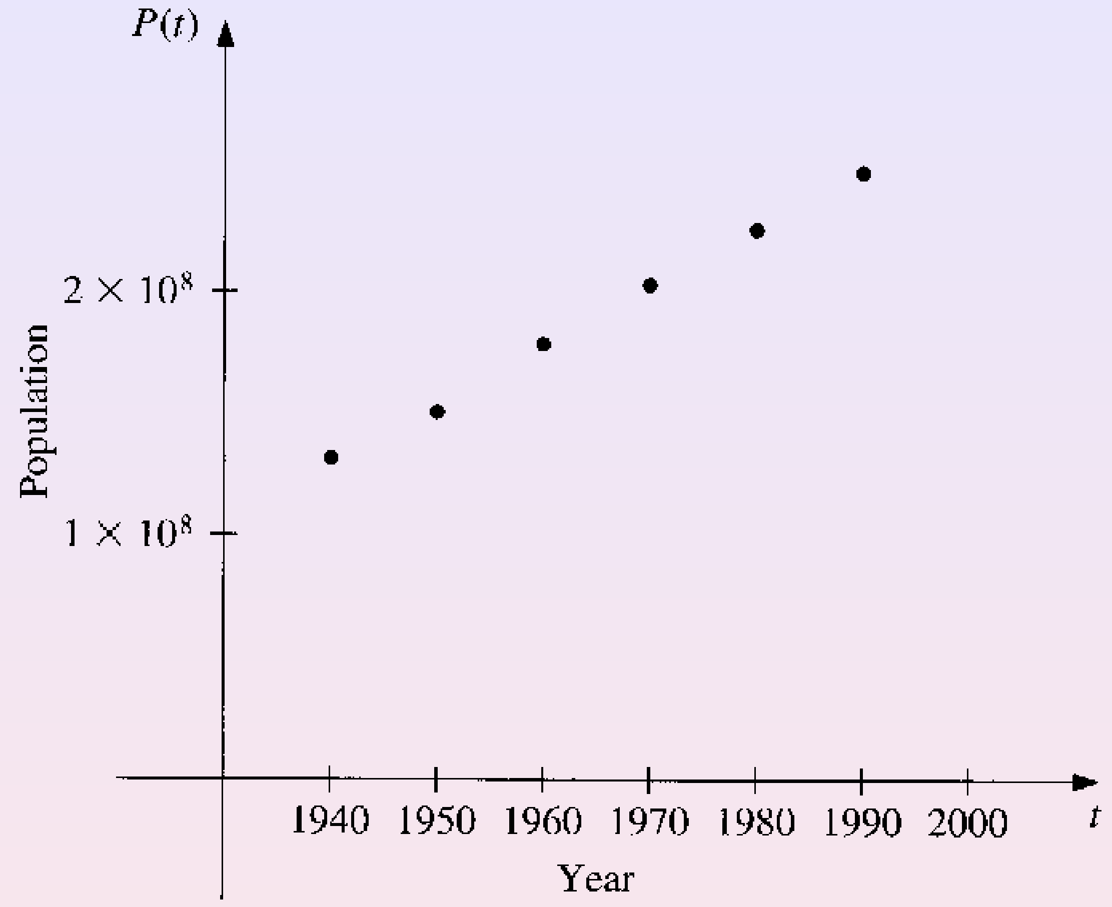
[1] These slides are based on Prof. Tsung-Ming Huang(NTNU)'s original slides

Outline

Introduction



Question

From these data, how do we get a reasonable estimate of the population, say, in 1965, or even in 2010?

Interpolation

Suppose we do not know the function f, but a few information (data) about f, now we try to compute a function g that approximates f.

Theorem (Weierstrass Approximation Theorem)

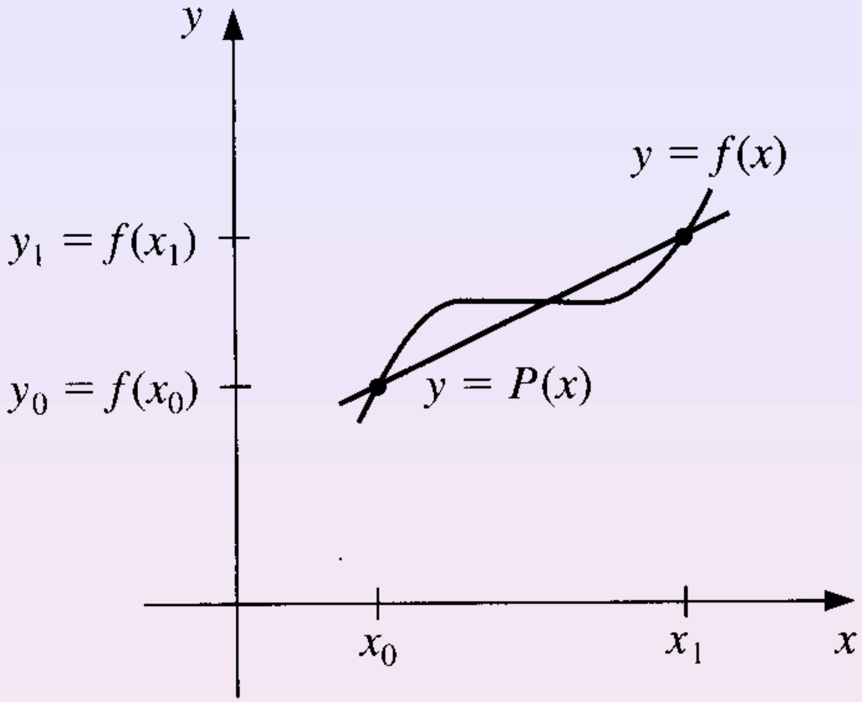
Suppose that f is defined and continuous on $[a, b]$. For any $\varepsilon > 0$, there exists a polynomial $P(x)$, such that

$$|f(x) - P(x)| < \varepsilon, \qquad \text{for all } x \text{ in } [a, b].$$

Reason for using polynomial

1. They uniformly approximate continuous functions (Weierstrass Theorem)
2. The derivatives and indefinite integral of a polynomial are easy to determine and are also polynomials.

Interpolation and the Lagrange polynomial



Property

The linear function passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is unique.

Let

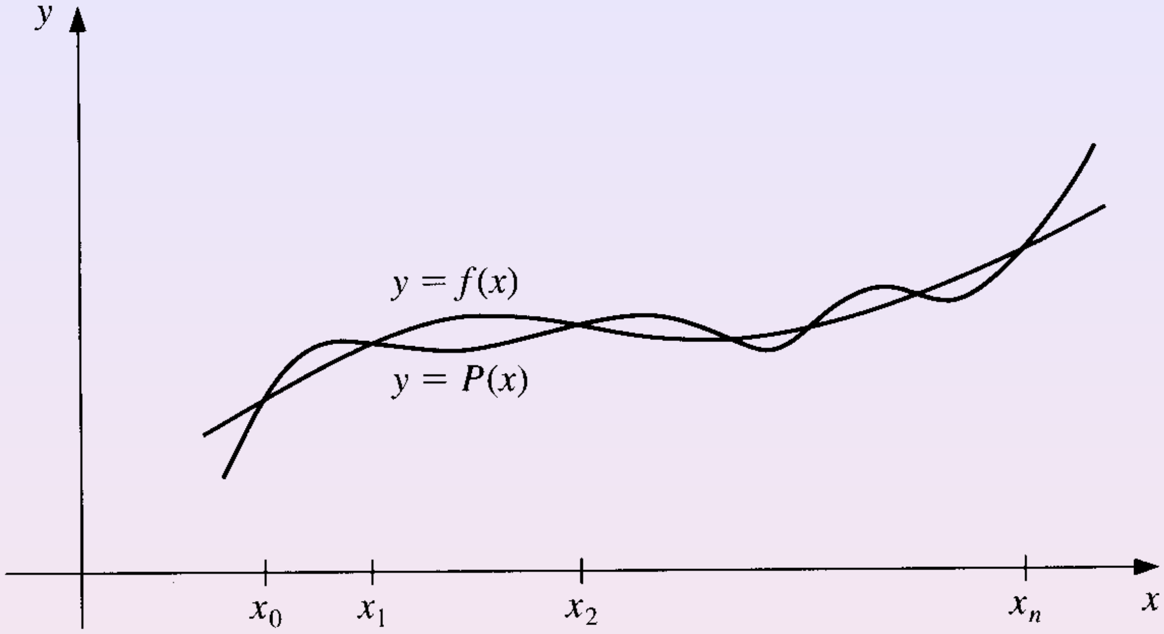
$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

and

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1).$$

Then

$$P(x_0) = f(x_0), \quad P(x_1) = f(x_1).$$



Question

How to find the polynomial of degree n that passes through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$?

Theorem

If (x_i, y_i), $x_i, y_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, *are* $n + 1$ *distinct pairs of data point, then there is a unique polynomial* P_n *of degree at most* n *such that*

$$P_n(x_i) = y_i, \quad (0 \leq i \leq n). \tag{1}$$

Proof of Existence (by mathematical induction)

1. The theorem clearly holds for $n = 0$ (only one data point (x_0, y_0)) since one may choose the constant polynomial $P_0(x) = y_0$ for all x.
2. Assume that the theorem holds for $n \leq k$, that is, there is a polynomial P_k, $\deg(P_k) \leq k$, such that $y_i = P_k(x_i)$, for $0 \leq i \leq k$.
3. Next we try to construct a polynomial of degree at most $k+1$ to interpolate (x_i, y_i), $0 \leq i \leq k+1$. Let

$$P_{k+1}(x) = P_k(x) + c(x - x_0)(x - x_1)\cdots(x - x_k),$$

where

$$c = \frac{y_{k+1} - P_k(x_{k+1})}{(x_{k+1} - x_0)(x_{k+1} - x_1)\cdots(x_{k+1} - x_k)}.$$

Since x_i are distinct, the polynomial $P_{k+1}(x)$ is well-defined and $\deg(P_{k+1}) \leq k+1$. It is easy to verify that

$$P_{k+1}(x_i) = y_i, \quad 0 \leq i \leq k+1.$$

Proof of Uniqueness

Suppose there are two such polynomials P_n and Q_n satisfying (1). Define

$$S_n(x) = P_n(x) - Q_n(x).$$

Since both deg$(P_n) \leq n$ and deg$(Q_n) \leq n$, deg$(S_n) \leq n$. Moreover

$$S_n(x_i) = P_n(x_i) - Q_n(x_i) = y_i - y_i = 0,$$

for $0 \leq i \leq n$. This means that S_n has at least $n+1$ zeros, it therefore must be $S_n = 0$. Hence $P_n = Q_n$. □

Idea

Construct polynomial $P(x)$ with deg$(P) \leq n$ as

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x),$$

where

1. $L_{n,k}(x)$ are polynomial with degree n for $0 \leq k \leq n$.
2. $L_{n,k}(x_k) = 1$ and $L_{n,k}(x_i) = 0$ for $i \neq k$.

Then

$$P(x_k) = f(x_k) \quad \text{for } k = 0, 1, \ldots, n.$$

1. deg$(L_{n,k}) = n$ and $L_{n,k}(x_i) = 0$ for $i \neq k$:

$$L_{n,k}(x) = c_k(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)$$

2. $L_{n,k}(x_k) = 1$:

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

Theorem

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

which is called the nth Lagrange interpolating polynomial.

Note that we will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

Example

Given the following 4 data points,

x_i	0	1	3	5
y_i	1	2	6	7

find a polynomial in Lagrange form to interpolate these datas.

Solution: The interpolating polynomial in the Lagrange form is

$$P_3(x) = L_0(x) + 2L_1(x) + 6L_2(x) + 7L_3(x) \quad \text{with}$$

$$\begin{aligned}
L_0(x) &= \frac{(x-1)(x-3)(x-5)}{(0-1)(0-3)(0-5)} = -\frac{1}{15}(x-1)(x-3)(x-5), \\
L_1(x) &= \frac{(x-0)(x-3)(x-5)}{(1-0)(1-3)(1-5)} = \frac{1}{8}x(x-3)(x-5), \\
L_2(x) &= \frac{(x-0)(x-1)(x-5)}{(3-0)(3-1)(3-5)} = -\frac{1}{12}x(x-1)(x-5), \\
L_3(x) &= \frac{(x-0)(x-1)(x-3)}{(5-0)(5-1)(5-3)} = \frac{1}{40}x(x-1)(x-3).
\end{aligned}$$

Question

What's the error involved in approximating $f(x)$ by the interpolating polynomial $P(x)$?

Theorem

Suppose

1. $x_0, \ldots, x_n$ *are distinct numbers in* $[a,b]$*,*
2. $f \in C^{n+1}[a,b]$.

Then, $\forall x \in [a,b]$, $\exists\ \xi(x) \in (a,b)$ *such that*

$$f(x) = P(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n). \quad (2)$$

Proof: If $x = x_k$, for any $k = 0, 1, \ldots, n$, then $f(x_k) = P(x_k)$ and (2) is satisfied.

If $x \neq x_k$, for all $k = 0, 1, \ldots, n$, define

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{t - x_i}{x - x_i}.$$

Since $f \in C^{n+1}[a, b]$ and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{n+1}[a, b]$. Since

$$g(x_k) = [f(x_k) - P(x_k)] - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x_k - x_i}{x - x_i} = 0,$$

and

$$g(x) = [f(x) - P(x)] - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x - x_i}{x - x_i} = 0,$$

it implies that g is zero at $x, x_0, x_1, \ldots, x_n$. By the Generalized Rolle's Theorem, $\exists\ \xi \in (a, b)$ such that $g^{n+1}(\xi) = 0$. That is

$$
\begin{aligned}
0 &= g^{(n+1)}(\xi) \qquad (3) \\
&= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi}.
\end{aligned}
$$

Since $\deg(P) \leq n$, it implies that $P^{(n+1)}(\xi) = 0$. On the other hand, $\prod_{i=0}^{n}[(t - x_i)/(x - x_i)]$ is a polynomial of degree $(n + 1)$, so

$$
\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^{n}(x - x_i)} \right] t^{n+1} + \text{(lower-degree terms in } t\text{)},
$$

and

$$
\frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right] = \frac{(n + 1)!}{\prod_{i=0}^{n}(x - x_i)}.
$$

Equation (3) becomes

$$0 = f^{(n+1)}(\xi) - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)},$$

i.e.,

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x - x_i).$$

□

Example

1. Goal: Prepare a table for the function $f(x) = e^x$ for $x \in [0, 1]$.
2. $x_{j+1} - x_j = h$ for $j = 0, 1, \ldots, n-1$.
3. What should h be for linear interpolation to give an absolute error of at most 10^{-6}?

Suppose $x \in [x_j, x_{j+1}]$. Equation (2) implies that

$$\begin{aligned}
|f(x) - P(x)| &= \left| \frac{f^{(2)}(\xi)}{2!}(x - x_j)(x - x_{j+1}) \right| \\
&= \frac{|f^{(2)}(\xi)|}{2!}|x - x_j||x - x_{j+1}| \\
&= \frac{e^{\xi}}{2}|x - jh||x - (j+1)h| \\
&\le \frac{1}{2}\left(\max_{\xi \in [0,1]} e^{\xi}\right)\left(\max_{x_j \le x \le x_{j+1}} |x - jh||x - (j+1)h|\right) \\
&= \frac{e}{2}\left(\max_{x_j \le x \le x_{j+1}} |x - jh||x - (j+1)h|\right).
\end{aligned}$$

Let

$$g(x) = (x - jh)(x - (j+1)h), \quad \text{for } jh \le x \le (j+1)h.$$

Then

$$\max_{x_j \le x \le x_{j+1}} |g(x)| = \left| g\left(\left(j + \frac{1}{2} \right) h \right) \right| = \frac{h^2}{4}.$$

Consequently,

$$|f(x) - P(x)| \le \frac{eh^2}{8} \le 10^{-6},$$

which implies that

$$h < 1.72 \times 10^{-3}.$$

Since $n = (1-0)/h$ must be an integer, one logical choice for the step size is $h = 0.001$.

Difficulty for the Lagrange interpolation

1. If more data points are added to the interpolation problem, all the cardinal functions L_k have to be recalculated.
2. We shall now derive the interpolating polynomials in a manner that uses the previous calculations to greater advantage.

Definition

1. f is a function defined at $x_0, x_1, \ldots, x_n$
2. Suppose that $m_1, m_2, \ldots, m_k$ are k distinct integers with $0 \le m_i \le n$ for each i.

The Lagrange polynomial that interpolates f at the k points $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$ is denoted $P_{m_1,m_2,\ldots,m_k}(x)$.

Theorem

Let f be defined at distinct points $x_0, x_1, \ldots, x_k$, and $0 \leq i, j \leq k$, $i \neq j$. Then

$$P(x) = \frac{(x - x_j)}{(x_i - x_j)} P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - \frac{(x - x_i)}{(x_i - x_j)} P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)$$

describes the k-th Lagrange polynomial that interpolates f at the $k+1$ points $x_0, x_1, \ldots, x_k$.

Proof: Since

$$\deg(P_{0,1,\ldots,j-1,j+1,\ldots,k}) \leq k - 1$$

and

$$\deg(P_{0,1,\ldots,i-1,i+1,\ldots,k}) \leq k - 1,$$

it implies that $\deg(P) \leq k$. If $0 \leq r \leq k$ and $r \neq i, j$, then

$$
\begin{aligned}
P(x_r) &= \frac{(x_r - x_j)}{(x_i - x_j)} P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_r) - \frac{(x_r - x_i)}{(x_i - x_j)} P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_r) \\
&= \frac{(x_r - x_j)}{(x_i - x_j)} f(x_r) - \frac{(x_r - x_i)}{(x_i - x_j)} f(x_r) = f(x_r).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
P(x_i) &= \frac{(x_i - x_j)}{(x_i - x_j)} P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_i) - \frac{(x_i - x_i)}{(x_i - x_j)} P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_i) \\
&= f(x_i)
\end{aligned}
$$

and

$$
\begin{aligned}
P(x_j) &= \frac{(x_j - x_j)}{(x_i - x_j)} P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_j) - \frac{(x_j - x_i)}{(x_i - x_j)} P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_j) \\
&= f(x_j).
\end{aligned}
$$

Therefore $P(x)$ agrees with f at all points $x_0, x_1, \ldots, x_k$. By the uniqueness theorem, $P(x)$ is the k-th Lagrange polynomial that interpolates f at the $k+1$ points $x_0, x_1, \ldots, x_k$, i.e., $P \equiv P_{0,1,\ldots,k}$.

Neville's method

The theorem implies that the Lagrange interpolating polynomial can be generated recursively. The procedure is called the Neville's method.

1. Denote
$$Q_{i,j} = P_{i-j,i-j+1,\ldots,i-1,i}.$$
2. Hence $Q_{i,j}$, $0 \le j \le i$, denotes the interpolating polynomial of degree j on the $j+1$ points $x_{i-j}, x_{i-j+1}, \ldots, x_{i-1}, x_i$.
3. The polynomials can be computed in a manner as shown in the following table.

x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

$$
\begin{array}{c|lllll}
x_0 & P_0 = Q_{0,0} & & & & \\
x_1 & P_1 = Q_{1,0} & P_{0,1} = Q_{1,1} & & & \\
x_2 & P_2 = Q_{2,0} & P_{1,2} = Q_{2,1} & P_{0,1,2} = Q_{2,2} & & \\
x_3 & P_3 = Q_{3,0} & P_{2,3} = Q_{3,1} & P_{1,2,3} = Q_{3,2} & P_{0,1,2,3} = Q_{3,3} & \\
x_4 & P_4 = Q_{4,0} & P_{3,4} = Q_{4,1} & P_{2,3,4} = Q_{4,2} & P_{1,2,3,4} = Q_{4,3} & P_{0,1,2,3,4} = Q_{4,4}
\end{array}
$$

$$
\begin{aligned}
P_{0,1}(x) &= \frac{(x-x_1)P_0(x) - (x-x_0)P_1(x)}{x_0 - x_1}, \\
P_{1,2}(x) &= \frac{(x-x_2)P_1(x) - (x-x_1)P_2(x)}{x_1 - x_2}, \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
P_{0,1,2}(x) &= \frac{(x-x_2)P_{0,1}(x) - (x-x_0)P_{1,2}(x)}{x_0 - x_2} \\
&\vdots
\end{aligned}
$$

Example

Compute approximate value of $f(1.5)$ by using the following datas:

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

1. The first-degree approximation:

$$
\begin{aligned}
Q_{1,1}(1.5) &= \frac{(x-x_0)Q_{1,0}-(x-x_1)Q_{0,0}}{x_1-x_0} \\
&= \frac{(1.5-1.0)Q_{1,0}-(1.5-1.3)Q_{0,0}}{1.3-1.0} = 0.5233449, \\
Q_{2,1}(1.5) &= \frac{(x-x_1)Q_{2,0}-(x-x_2)Q_{1,0}}{x_2-x_1} \\
&= \frac{(1.5-1.3)Q_{2,0}-(1.5-1.6)Q_{1,0}}{1.6-1.3} = 0.5102968, \\
Q_{3,1}(1.5) &= \frac{(x-x_2)Q_{3,0}-(x-x_3)Q_{2,0}}{x_3-x_2} \\
&= \frac{(1.5-1.6)Q_{3,0}-(1.5-1.9)Q_{2,0}}{1.9-1.6} = 0.5132634, \\
Q_{4,1}(1.5) &= \frac{(x-x_3)Q_{4,0}-(x-x_4)Q_{3,0}}{x_4-x_3} \\
&= \frac{(1.5-1.9)Q_{4,0}-(1.5-2.2)Q_{3,0}}{2.2-1.9} = 0.5104270.
\end{aligned}
$$

1. The second-degree approximation:

$$
\begin{aligned}
Q_{2,2}(1.5) &= \frac{(x - x_1)Q_{2,1} - (x - x_2)Q_{1,1}}{x_2 - x_1} \\
&= \frac{(1.5 - 1.3)Q_{2,1} - (1.5 - 1.6)Q_{1,1}}{1.6 - 1.3} = 0.5124715, \\
Q_{3,2}(1.5) &= \frac{(x - x_2)Q_{3,1} - (x - x_3)Q_{2,1}}{x_3 - x_2} \\
&= \frac{(1.5 - 1.6)Q_{3,1} - (1.5 - 1.9)Q_{2,1}}{1.9 - 1.6} = 0.5112857, \\
Q_{4,2}(1.5) &= \frac{(x - x_3)Q_{4,1} - (x - x_4)Q_{3,1}}{x_4 - x_3} \\
&= \frac{(1.5 - 1.9)Q_{4,1} - (1.5 - 2.2)Q_{3,1}}{2.2 - 1.9} = 0.5137361.
\end{aligned}
$$

x	$f(x)$	1st-deg	2nd-deg	3rd-deg	4th-deg
1.0	0.7651977				
1.3	0.6200860	0.5233449			
1.6	0.4554022	0.5102968	0.5124715		
1.9	0.2818186	0.5132634	0.5112857	0.5118127	
2.2	0.1103623	0.5104270	0.5137361	0.5118302	0.5118200

Table: Results of the higher-degree approximations

x	$f(x)$	1st-deg	2nd-deg	3rd-deg	4th-deg	5th-deg
1.0	0.7651977					
1.3	0.6200860	0.5233449				
1.6	0.4554022	0.5102968	0.5124715			
1.9	0.2818186	0.5132634	0.5112857	0.5118127		
2.2	0.1103623	0.5104270	0.5137361	0.5118302	0.5118200	
2.5	−0.0483838	0.4807699	0.5301984	0.5119070	0.5118430	0.5118277

Table: Results of adding $(x_5, f(x_5))$