Hermite Interpolation

Given $n+1$ data points $x_0 < x_1 < \cdots < x_n$, and

$$\begin{array}{cccc}
y_0^{(0)} = f(x_0) & y_1^{(0)} = f(x_1) & \cdots & y_n^{(0)} = f(x_n) \\
y_0^{(1)} = f'(x_0) & y_1^{(1)} = f'(x_1) & \cdots & y_n^{(1)} = f'(x_n) \\
\vdots & \vdots & & \vdots \\
y_0^{(m_0)} = f^{(m_0)}(x_0) & y_1^{(m_1)} = f^{(m_1)}(x_1) & \cdots & y_n^{(m_n)} = f^{(m_n)}(x_n) \\
\downarrow & \downarrow & & \downarrow \\
m_0 + 1 \text{ conditions} & m_1 + 1 \text{ conditions} & \cdots & m_n + 1 \text{ conditions}
\end{array}$$

for some function $f \in C^m[a,b]$, where $m = \max\{m_0, m_1, \ldots, m_n\}$.

- Determine a polynomial P of degree at most N, where

$$N = \left(\sum_{i=0}^{n} m_i\right) + n, \tag{5}$$

to satisfy the following interpolation conditions:

$$P^{(k)}(x_i) = y_i^{(k)}, \qquad k = 0, 1, \ldots m_i, \quad i = 0, 1, \ldots, n. \tag{6}$$

- If $n = 0$, then P is the m_0th Taylor polynomial for f at x_0.
- If $m_i = 0$ for each i, then P is the nth Lagrange polynomial interpolating f on $x_0, \ldots, x_n$.
- If $m_i = 1$ for each i, then P is called the Hermite polynomial.

Theorem

If $f \in C^1[a,b]$ *and* $x_0, \ldots, x_n \in [a,b]$ *are distinct, then the polynomial of least degree agreeing with* f *and* f' *at* $x_0, \ldots, x_n$ *is unique and is given by*

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \widehat{H}_{n,j}(x),$$

where

$$H_{n,j}(x) = \left[1 - 2(x - x_j) L'_{n,j}(x_j)\right] L^2_{n,j}(x), \quad \widehat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x),$$

and

$$L_{n,j}(x) = \frac{(x - x_0) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)}.$$

Moreover, if $f \in C^{2n+2}[a,b]$*, then* $\exists\, \xi(x) \in [a,b]$ *s.t.*

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x)).$$

Proof: The representation

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\widehat{H}_{n,j}(x),$$

suggests that it suffices to construct $H_{n,j}(x)$ and $\hat{H}_{n,j}(x)$ with

$$\begin{cases} H_{n,j}(x_j) = 1 \\ H'_{n,j}(x_j) = 0 \end{cases}, \quad H_{n,j}(x_i) = H'_{n,j}(x_i) = 0 \quad \text{if } i \neq j,$$

and

$$\begin{cases} \hat{H}_{n,j}(x_j) = 0 \\ \hat{H}'_{n,j}(x_j) = 1 \end{cases}, \quad \hat{H}_{n,j}(x_i) = \hat{H}'_{n,j}(x_i) = 0 \quad \text{if } i \neq j,$$

It is easy to see that $\deg H_{n,j} \leq 2n+1$ and $\deg H_{n,j} \leq 2n+1$.
Since $\deg L_{n,j}^2 = 2n$ and

$$L_{n,j}^2(x_i) = (L_{n,j}^2)'(x_i) = 0, \text{for } i \neq j$$

We can simply seek for $H_{n,j}(x)$ and $\hat{H}_{n,j}(x)$ of the form

$$H_{n,j}(x) = (a(x-x_j)+b)L_{n,j}^2(x), \quad \hat{H}_{n,j}(x) = (\hat{a}(x-x_j)+\hat{b})L_{n,j}^2(x)$$

The coefficients a, b and $\hat{a}$, $\hat{b}$ can be easily solved from the conditions

$$H_{n,j}(x_i) = 1, \quad H'_{n,j}(x_i) = 0,$$

and

$$\hat{H}_{n,j}(x_i) = 0, \quad \hat{H}'_{n,j}(x_i) = 1,$$

respectively. □

Proof of uniqueness:

- Since $\deg(P) \le 2n+1$, write

$$P(x) = a_0 + a_1 x + \cdots + a_{2n+1} x^{2n+1}.$$

- $2n+2$ coefficients, $a_0, a_1, \ldots, a_{2n+1}$, to be determined and $2n+2$ conditions given

$$P(x_i) = f(x_i), \quad P'(x_i) = f'(x_i), \quad \text{for } i = 0, \ldots, n.$$

$\Rightarrow$ $2n+2$ linear equations in $2n+2$ unknowns to solve
$\Rightarrow$ show that the coefficient matrix A of this system is nonsingular.

- To prove A is nonsingular, it suffices to prove that $Au = 0$ has only the trivial solution $u = 0$.
- $Au = 0$ iff

$$P(x_i) = 0, \quad P'(x_i) = 0, \quad \text{for } i = 0, \ldots, n.$$

$\Rightarrow P$ is a multiple of the polynomial given by

$$q(x) = \prod_{i=0}^{n} (x - x_i)^2 .$$

- However, $\deg(q) = 2n + 2$ whereas P has degree at most N.
- Therefore, $P(x) = 0$, i.e. $u = 0$.
- That is, A is nonsingular, and the Hermite interpolation problem has a unique solution. □

Divided Difference Method for Hermite Interpolation

Given the $2n+2$ condition pairs

$$(x_0, f(x_0)), (x_0, f'(x_0)), (x_1, f(x_1)), (x_1, f'(x_1)), \cdots, (x_n, f(x_n)), (x_n, f'(x_n$$

Rename the x-coordinates as $z_0, z_1, \cdots, z_{2n+1}$, where

$$z_0 = z_1 = x_0, z_2 = z_3 = x_1, \cdots, z_{2n+1} = z_{2n+2} = x_n.$$

Note that $z_0 \le z_1 \le \cdots \le z_N$. If z_j were distinct, then the unique Hermite interpolating polynomial in Newton's form is given by

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+2} f[z_0, z_1, \ldots, z_k](x - z_0)(x - z_1)\cdots(x - z_{k-1}).$$

When $k \ge 2$, $z_i \neq z_{i+k}$, the k-th divided difference is well defined:

$$f[z_i, z_{i+1}, \ldots, z_{i+k}] = \frac{f[z_{i+1}, z_{i+2}, \ldots, z_{i+k}] - f[z_i, z_{i+1}, \ldots, z_{i+k-1}]}{z_{i+k} - z_i}.$$

However the first divided-difference formula has to be modified since $z_{2i} = z_{2i+1} = x_i$ for each i. Let

$$z_{2i} = x_i, \qquad z_{2i+1}^{\epsilon} = x_i + \epsilon.$$

and let $\epsilon \to 0$. Formally, it suffices to replace the first divided differences by

$$f[z_{2i}, z_{2i+1}] := \lim_{\epsilon \to 0} f[z_{2i}, z_{2i+1}^{\epsilon}] = f'(z_{2i}) = f'(x_i)$$

z	$f(z)$		
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1^\epsilon] = \frac{f[z_1^\epsilon]-f[z_0]}{z_1^\epsilon - z_0}$	
$z_1^\epsilon = x_0 + \epsilon$	$f[z_1^\epsilon] = f(z_1^\epsilon)$		$f[z_0, z_1^\epsilon, z_2] = \frac{f[z_1^\epsilon, z_2]-f[z_0, z_1^\epsilon]}{z_2 - z_0}$
		$f[z_1^\epsilon, z_2] = \frac{f[z_2]-f[z_1^\epsilon]}{z_2 - z_1^\epsilon}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1^\epsilon, z_2, z_3^\epsilon] = \frac{f[z_2, z_3^\epsilon]-f[z_1^\epsilon, z_2]}{z_3^\epsilon - z_1^\epsilon}$
		$f[z_2, z_3^\epsilon] = \frac{f[z_3^\epsilon]-f[z_2]}{z_3^\epsilon - z_2}$	
$z_3^\epsilon = x_1 + \epsilon$	$f[z_3^\epsilon] = f(z_3^\epsilon)$		$f[z_2, z_3^\epsilon, z_4] = \frac{f[z_3^\epsilon, z_4]-f[z_2, z_3^\epsilon]}{z_4 - z_2}$
		$f[z_3^\epsilon, z_4] = \frac{f[z_4]-f[z_3^\epsilon]}{z_4 - z_3^\epsilon}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3^\epsilon, z_4, z_5^\epsilon] = \frac{f[z_4, z_5^\epsilon]-f[z_3^\epsilon, z_4]}{z_5^\epsilon - z_3^\epsilon}$
		$f[z_4, z_5^\epsilon] = \frac{f[z_5^\epsilon]-f[z_4]}{z_5^\epsilon - z_4}$	
$z_5^\epsilon = x_2 + \epsilon$	$f[z_5^\epsilon] = f(z_5^\epsilon)$		

As $\epsilon \to 0$, $z_1^\epsilon \to z_1 := x_0$, $f[z_1^\epsilon] \to f(x_0)$, $f[z_0, z_1^\epsilon] \to f[z_0, z_1] := f'(x_0)$, etc.

z	$f(z)$		
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \frac{f[z_1,z_2]-f[z_0,z_1]}{z_2-z_0}$
		$f[z_1, z_2] = \frac{f[z_2]-f[z_1]}{z_2-z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \frac{f[z_2,z_3]-f[z_1,z_2]}{z_3-z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \frac{f[z_3,z_4]-f[z_2,z_3]}{z_4-z_2}$
		$f[z_3, z_4] = \frac{f[z_4]-f[z_3]}{z_4-z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \frac{f[z_4,z_5]-f[z_3,z_4]}{z_5-z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+2} f[z_0, z_1, \ldots, z_k](x - z_0)(x - z_1)\cdots(x - z_{k-1}).$$