Divided Differences

- Neville's method: generate successively higher-degree polynomial approximations at a specific point.
- Divided-difference methods: successively generate the polynomials themselves.

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with f at distinct $x_0, x_1, \ldots, x_n$. Express $P_n(x)$ in the form

$$\begin{aligned} P_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ &\quad + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

- Determine constant a_0:

$$a_0 = P_n(x_0) = f(x_0)$$

- Determine a_1:

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1) \quad \Rightarrow \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

- The zero divided difference of the function f with respect to x_i:

$$f[x_i] = f(x_i)$$

- The first divided difference of f with respect to x_i and x_{i+1} is denoted and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

- The second divided difference of f is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

- The k-th divided difference relative to $x_i, x_{i+1}, \ldots, x_{i+k}$ is given by

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

As might be expected from the evaluation of $a_0, a_1, \ldots, a_n$ in $P_n(x)$, the required coefficients are given by

$$a_k = f[x_0, x_1, \ldots, x_k], \qquad k = 0, 1, \ldots, n.$$

Therefore $P_n(x)$ can be expressed as

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1) \cdots (x - x_{k-1}).$$

This formula is known as the Newton's divided-difference formula.

x_i	$k=0$	$k=1$	$k=2$	$k=3$
x_0	$f[x_0]$			
		$> \quad f[x_0, x_1]$		
x_1	$f[x_1]$		$> \quad f[x_0, x_1, x_2]$	
		$> \quad f[x_1, x_2]$		$> \quad f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$> \quad f[x_1, x_2, x_3]$	
		$> \quad f[x_2, x_3]$		
x_3	$f[x_3]$			

Table: Dependency diagram of Newton's divided differences.

Newton's Divided Difference

Step 0 INPUT $(x_0, y_0), \cdots, (x_n, y_n)$. Set $F_{0,0} = y_0, \cdots, F_{n,0} = y_n$.

Step 1. For $i = 1, 2, \cdots, n$

For $k = 1, 2, \cdots, i$

$$F_{i,k} = \frac{F_{i,k-1} - F_{i-1,k-1}}{x_i - x_{i-k}} \quad (F_{i,k} = f[x_{i-k}, \cdots, x_k])$$

End k

End i

Step 2. OUTPUT $F_{0,0}, F_{1,1}, \cdots, F_{n,n}$ $(F_{i,i} = f[x_0, \cdots, x_i])$, STOP.

x_i	$k = 0$	$k = 1$	$k = 2$	$k = 3$
x_0	$f[x_0]$			
		$> \; f[x_0, x_1]$		
x_1	$f[x_1]$		$> \; f[x_0, x_1, x_2]$	
		$> \; f[x_1, x_2]$		$> \; f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$> \; f[x_1, x_2, x_3]$	
		$> \; f[x_2, x_3]$		
x_3	$f[x_3]$			

Table: Divided differences generated from left to right, then from top to bottom.
Red: $i = 0$; green: $i = 1$; blue: $i = 2$; black: $i = 3$.

Step 0 INPUT $(x_0, y_0), \cdots, (x_n, y_n)$. Set $F_0 = y_0, \cdots, F_n = y_n$.

Step 1. For $k = 1, 2, \cdots, n$

For $i = n, n-1, \cdots, k$

$$F_i \leftarrow \frac{F_i - F_{i-1}}{x_i - x_{i-k}} \quad \left(F_{i,k} = \frac{F_{i,k-1} - F_{i-1,k-1}}{x_i - x_{i-k}} \right)$$

End i

End k

Step 2. OUTPUT $F_0, F_1, \cdots, F_n$ $(F_i = f[x_0, \cdots, x_i])$, STOP.

x_i	$k=0$		$k=1$		$k=2$		$k=3$
x_0	$f[x_0]$						
		$>$	$f[x_0, x_1]$				
x_1	$f[x_1]$			$>$	$f[x_0, x_1, x_2]$		
		$>$	$f[x_1, x_2]$			$>$	$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$			$>$	$f[x_1, x_2, x_3]$		
		$>$	$f[x_2, x_3]$				
x_3	$f[x_3]$						

Table: Divided differences generated from bottom to top, then from left to right.

Example

Given the following 4 points ($n = 3$)

x_i	0	1	3	5
y_i	1	2	6	7

find a polynomial of degree 3 in Newton's form to interpolate these data.

Solution:

x_i	$k=0$	$k=1$	$k=2$	$k=3$
0	1			
		> 1		
1	2		$> \frac{1}{3}$	
		> 2		$> -\frac{17}{120}$
3	6		$> -\frac{3}{8}$	
		$> \frac{1}{2}$		
5	7			

Therefore, $P(x) = 1 + x + \dfrac{1}{3}x(x-1) - \dfrac{17}{120}x(x-1)(x-3)$.

Theorem (11)

Suppose $f \in C^n[a,b]$ *and* $x_0, x_1, \ldots, x_n$ *are distinct numbers in* $[a,b]$. *Then there exists* $\xi \in (a,b)$ *such that*

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof: Define

$$g(x) = f(x) - P_n(x).$$

Since $P_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$, g has $n+1$ distinct zeros in $[a,b]$. By the generalized Rolle's Theorem, $\exists\ \xi \in (a,b)$ such that

$$0 = g^{(n)}(\xi) = f^{(n)}(\xi) - P_n^{(n)}(\xi).$$

Note that

$$P_n^{(n)}(x) = n!\, f[x_0, x_1, \ldots, x_n].$$

As a consequence

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Let

$$h = \frac{x_n - x_0}{n} = x_{i+1} - x_i, \qquad i = 0, 1, \ldots, n-1.$$

Then each $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$. For any $x \in [a, b]$, write

$$x = x_0 + sh, \qquad s \in \mathbb{R}.$$

Hence $x - x_i = (s - i)h$ and

$$\begin{aligned}
P_n(x) &= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1}) \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](s - 0)h(s - 1)h\cdots(s - k + 1)h \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k]s(s - 1)\cdots(s - k + 1)h^k \\
&= f(x_0) + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k]k!\binom{s}{k}h^k, \qquad (4)
\end{aligned}$$

- The binomial formula

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$$

- The forward difference notation $\triangle$

$$\triangle f(x_i) = f(x_{i+1}) - f(x_i)$$

and for $i = 0, 1, \ldots, n-1$,

$$\triangle^k f(x_i) = \triangle^{k-1} f(x_{i+1}) - \triangle^{k-1} f(x_i) = \triangle \left(\triangle^{k-1} f(x_i) \right).$$

- With this notation,

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \triangle f(x_0) \\ f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{\frac{1}{h}\triangle f(x_1) - \frac{1}{h}\triangle f(x_0)}{2h} = \frac{1}{2h^2} \triangle^2 f(x_0), \end{aligned}$$

- In general

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!\, h^k} \triangle^k f(x_0).$$

- The Newton forward-difference formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0).$$

- If x is close to x_0, the first few terms in the above summation also gives good lower degree interpolating polynomials.

- In case x is close to x_n, we could rearrange interpolation nodes as $x_n, x_{n-1}, \ldots, x_0$:

$$\begin{aligned} P_n(x) &= f[x_n] + f[x_n, x_{n-1}](x - x_n) \\ &\quad + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1}) + \cdots \\ &\quad + f[x_n, \ldots, x_0](x - x_n)(x - x_{n-1}) \cdots (x - x_1). \end{aligned}$$

- If the nodes are equally spaced with

$$h = \frac{x_n - x_0}{n}, \qquad x_i = x_n - (n - i)h, \qquad x = x_n + sh,$$

then

$$\begin{aligned} P_n(x) &= f[x_n] + \sum_{k=1}^{n} f[x_n, x_{n-1}, \ldots, x_{n-k}] sh(s+1)h \cdots (s+k-1)h \\ &= f(x_n) + \sum_{k=1}^{n} f[x_n, x_{n-1}, \ldots, x_{n-k}](-1)^k k! \binom{-s}{k} h^k \end{aligned}$$

- The first few terms in the above summation gives good lower degree approximation for x near x_n.

- Binomial formula:

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}.$$

- Backward-difference:

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1}) = \nabla\left(\nabla^{k-1} f(x_i)\right),$$

then

$$f[x_n, x_{n-1}] = \frac{1}{h}\nabla f(x_n), \qquad f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2}\nabla^2 f(x_n),$$

and, in general,

$$f[x_n, x_{n-1}, \ldots, x_{n-k}] = \frac{1}{k!\,h^k}\nabla^k f(x_n).$$

- Newton backward-difference formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_k).$$