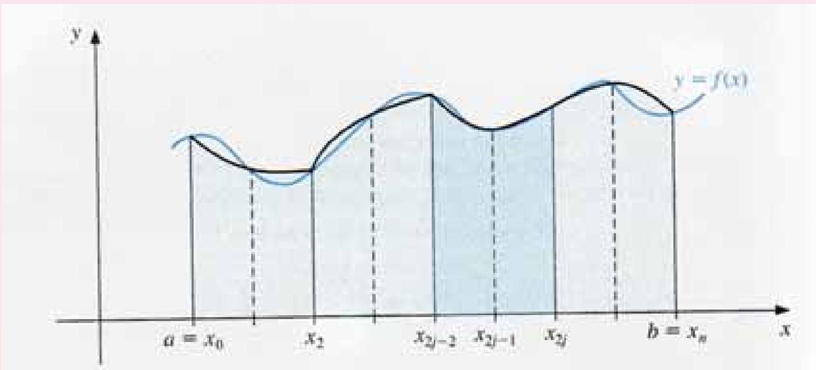
Composite Numerical Integration

- The Newton-Cotes formulas are generally not suitable for numerical integration over large interval. Higher degree formulas would be required, and the coefficients in these formulas are difficult to obtain.
- Also the Newton-Cotes formulas which are based on polynomial interpolation would be inaccurate over a large interval because of the oscillatory nature of high-degree polynomials.
- Now we discuss a piecewise approach, called composite rule, to numerical integration over large interval that uses the low-order Newton-Cotes formulas.
  - A composite rule is one obtained by applying an integration formula for a single interval to each subinterval of a partitioned interval.



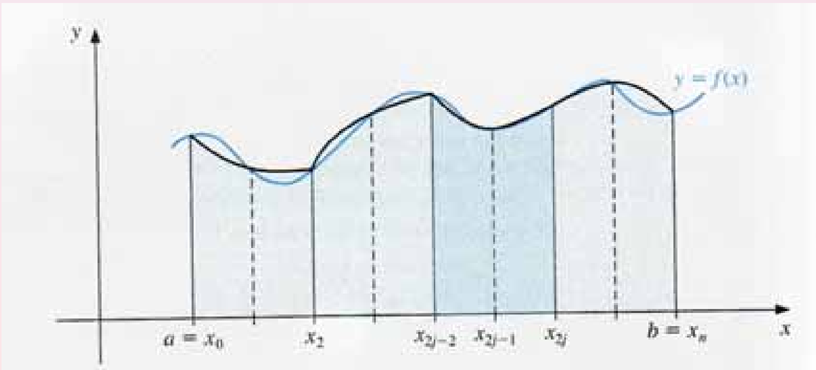
To illustrate the procedure, we choose an even integer n and partition the interval $[a, b]$ into n subintervals by nodes $a = x_0 < x_1 < \cdots < x_n = b$, and apply Simpson's rule on each consecutive pair of subintervals. With

$$h = \frac{b-a}{n} \quad \text{and} \quad x_j = a + jh, \quad j = 0, 1, \ldots, n,$$

we have on each interval $[x_{2j-2}, x_{2j}]$,

$$\int_{x_{2j-2}}^{x_{2j}} f(x)\,dx = \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \frac{h^5}{90} f^{(4)}(\xi_j),$$

for some $\xi_j \in (x_{2j-2}, x_{2j})$, provided that $f \in C^4[a, b]$.



The composite rule is obtained by summing up over the entire interval, that is,

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx \\
&= \sum_{j=1}^{n/2} \left[\frac{h}{3} \left(f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) \right) - \frac{h^5}{90} f^{(4)}(\xi_j) \right] \\
&= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2) \\
&\quad + f(x_2) + 4f(x_3) + f(x_4) \\
&\quad + f(x_4) + 4f(x_5) + f(x_6) \\
&\quad \vdots \\
&\quad + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)
\end{aligned}
$$

Hence

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{h}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ &\quad + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big] - \frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) \\ &= \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(x_n)\right] \\ &\quad -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j).\end{aligned}$$

To estimate the error associated with approximation, since $f \in C^4[a,b]$, we have, by the Extreme Value Theorem,

$$\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x),$$

for each $\xi_j \in (x_{2j-2}, x_{2j})$.

Hence

$$\frac{n}{2} \min_{x\in[a,b]} f^{(4)}(x) \leq \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \frac{n}{2} \max_{x\in[a,b]} f^{(4)}(x),$$

and

$$\min_{x\in[a,b]} f^{(4)}(x) \leq \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x).$$

By the Intermediate Value Theorem, there exists $\mu \in (a,b)$ such that

$$f^{(4)}(\mu) = \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j).$$

Thus, by replacing $n = (b-a)/h$,

$$\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = \frac{n}{2} f^{(4)}(\mu) = \frac{b-a}{2h} f^{(4)}(\mu).$$

Consequently, the composite Simpson's rule is derived.

Composite Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(b)\right] - \frac{b-a}{180} f^{(4)}(\mu)h^4,$$

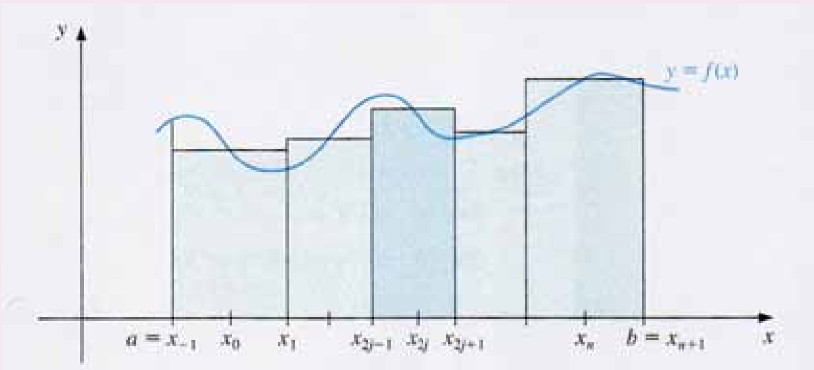
where n is an even integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \ldots, n$, and some $\mu \in (a, b)$.

The composite Midpoint rule can be derived in a similar way, except the midpoint rule is applied on each subinterval $[x_{2j-1}, x_{2j+1}]$ instead. That is,

$$\int_{x_{2j-1}}^{x_{2j+1}} f(x)\,dx = 2hf(x_{2j}) + \frac{h^3}{3} f''(\xi_j), \qquad j = 1, 2, \ldots, \frac{n}{2}.$$

Note that n must again be even. Consequently,

$$\int_a^b f(x)\,dx = 2h \sum_{j=1}^{n/2} f(x_{2j}) + \frac{h^3}{3} \sum_{j=1}^{n/2} f''(\xi_j).$$



The error term can be written as

$$\sum_{j=1}^{n/2} f''(\xi_j) = \frac{n}{2} f''(\mu) = \frac{b-a}{2h} f''(\mu),$$

for some $\mu \in (a, b)$.

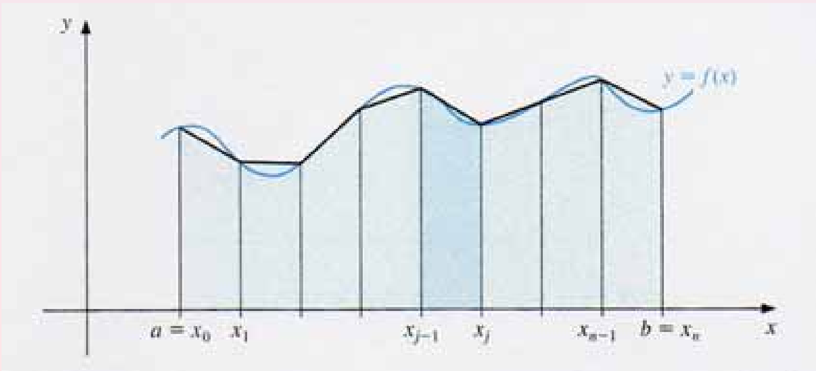
Composite Midpoint Rule

$$\int_a^b f(x)\,dx = 2h \sum_{j=1}^{n/2} f(x_{2j}) + \frac{b-a}{6} f''(\mu) h^2, \qquad (29)$$

where n is an even integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \ldots, n$, and some $\mu \in (a, b)$.

To derive the composite Trapezoidal rule, we partition $[a,b]$ by n equally spaced nodes $a = x_0 < x_1 < \cdots < x_n = b$, where n can be either odd or even. Apply the trapezoidal rule on $[x_{j-1}, x_j]$ and sum them up to obtain

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx \\ &= \sum_{j=1}^{n} \left\{ \frac{h}{2}\left[f(x_{j-1}) + f(x_j)\right] - \frac{h^3}{12} f''(\xi_j) \right\} \\ &= \frac{h}{2}\{[f(x_0)+f(x_1)] + [f(x_1)+f(x_2)] + \cdots \\ &\quad + [f(x_{n-1}) + f(x_n)]\} - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)\end{aligned}$$



Hence,

$$\begin{aligned}
\int_a^b f(x)\,dx &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \\
&\quad - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j) \\
&= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j) \\
&= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} f''(\mu) h^2,
\end{aligned}$$

where each $\xi_j \in (x_{j-1}, x_j)$ and $\mu \in (a, b)$.

Composite Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} f''(\mu)h^2, \qquad (30)$$

where n is an integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \ldots, n$, and some $\mu \in (a, b)$.