

Numerical Integration

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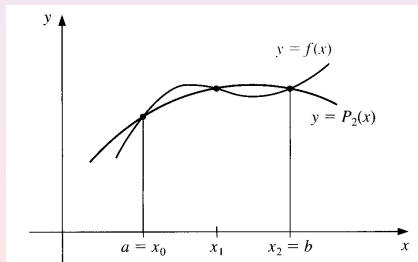
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If we choose $x_0 = a$, $x_1 = \frac{1}{2}(a + b)$, $x_2 = b$, $h = (b - a)/2$, and the second order Lagrange polynomial to interpolate $f(x)$, then

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx + \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)(x - x_2)}{6} f^{(3)}(\zeta(x)) dx.$$



It holds that

$$\int_a^b f(x) dx = h \left[\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2) \right] + \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)(x-x_2)}{6} f^{(3)}(\zeta(x)) dx.$$

It can show that there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{f^{(4)}(\xi)}{90} h^5.$$

This gives the **Simpson's rule formulation**.



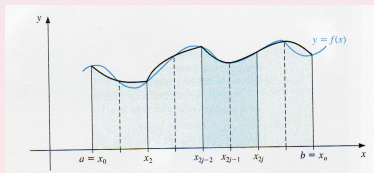
To illustrate the procedure, we choose an even integer n and partition the interval $[a, b]$ into n subintervals by nodes $a = x_0 < x_1 < \cdots < x_n = b$, and apply Simpson's rule on each consecutive pair of subintervals. With

$$h = \frac{b - a}{n} \quad \text{and} \quad x_j = a + jh, \quad j = 0, 1, \dots, n,$$

we have on each interval $[x_{2j-2}, x_{2j}]$,

$$\int_{x_{2j-2}}^{x_{2j}} f(x) dx = \frac{h}{3} [f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{h^5}{90} f^{(4)}(\xi_j),$$

for some $\xi_j \in (x_{2j-2}, x_{2j})$, provided that $f \in C^4[a, b]$.



The composite rule is obtained by summing up over the entire interval, that is,

$$\begin{aligned}
 \int_a^b f(x) dx &= \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) dx \\
 &= \sum_{j=1}^{n/2} \left[\frac{h}{3} (f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})) - \frac{h^5}{90} f^{(4)}(\xi_j) \right] \\
 &= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2) \\
 &\quad + f(x_2) + 4f(x_3) + f(x_4) \\
 &\quad + f(x_4) + 4f(x_5) + f(x_6) \\
 &\quad \vdots \\
 &\quad + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)
 \end{aligned}$$



Hence

$$\begin{aligned}\int_a^b f(x) dx &= \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ &\quad + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \\ &= \frac{h}{3} \left[f(x_0) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(x_n) \right] \\ &\quad - \frac{b-a}{180} f^{(4)}(\mu) h^4,\end{aligned}$$

for some $\mu \in (a, b)$.



Composite Simpson's Rule

$$\int_a^b f(x) dx = \frac{h}{3} \left[f(a) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(b) \right] - \frac{b-a}{180} f^{(4)}(\mu) h^4,$$

where n is an even integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \dots, n$, and some $\mu \in (a, b)$.

