1. (10 pts) Let $S = \text{span}\{(1,2,3)^T\}$, find a basis for $S^\perp$

Ans: a basis for $S^\perp = \left\{ \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$ ← 沒寫 basis 的量也扣 3 分

or " $= \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$, (not unique.)

2. (20 pts) Given a set of data $(x,y) = \{(-1,0), (0,1), (1,3), (2,5)\}$

(a) Find the best least square fit by a linear function.

(b) Find the best least square fit by a quadratic function.

Sol:

(a) Consider $y = c_1 + c_0 x$, then $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

$\Rightarrow c_1 = \frac{7}{5}$, $c_0 = \frac{17}{10}$.

The best least square fit by a linear function: $y = \frac{7}{5} + \frac{17}{10}x$.

(b) Consider $y = c_2 + c_1 x + c_0 x^2$, then $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

$\Rightarrow c_2 = \frac{23}{20}$, $c_1 = \frac{29}{20}$, $c_0 = \frac{1}{4}$.

The best least square fit by a quadratic function: $y = \frac{23}{20} + \frac{29}{20}x + \frac{1}{4}x^2$.

(給分標準): ① 沒寫出來以來，一律扣 2 分

② 計算錯誤，扣 5 分.

(3) (25 pts) $A=\begin{pmatrix}1&1&1\\0&1&0\\1&0&1\end{pmatrix}$, $b=\begin{pmatrix}1\\1\\1\end{pmatrix}$

10% (a) Find a basis for $N(A)$, $R(A)$, $N(A^T)$ and $R(A^T)$, respectively.

5% (b) Is $b \in R(A)$? Explain.

10% (c) Find $y_0 \in R(A)$ s.t. $\|b-y_0\| \le \|b-y\|$ for all $y \in R(A)$.

Sol:

(a)

$R(A)=\text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}=\text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\}$.

Note that $\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\}$ is linearly independent, so we know

a basis for $R(A)=\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\}$.

$N(A)=\{x \mid Ax=0\}$. Consider $\begin{pmatrix}1&1&1\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, where $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

Then $x=\begin{pmatrix}t\\0\\-t\end{pmatrix}$, $t \in \mathbb{R}$.

So, a basis for $N(A)=\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\}$.

$R(A^T)=\text{span}\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}=\text{span}\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}$.

Note that $\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}$ is linearly independent, so we know

a basis for $R(A^T)=\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}$.

$N(A^T)=\{x \mid A^Tx=0\}$. Consider $\begin{pmatrix}1&0&1\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, where $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

Then $x=\begin{pmatrix}t\\-t\\-t\end{pmatrix}$, $t \in \mathbb{R}$

So, a basis for $N(A^T)=\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\}$.

(給分標準): ① 這四了部分，錯一題就扣2分.

② $R(A)$, $N(A)$: 扣計5分.

$R(A^T)$, $N(A^T)$: 扣計5分.

(b) No.

If it was, i.e., $b=\begin{pmatrix}1\\1\\1\end{pmatrix} \in \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = R(A)$.

$\Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}\alpha+\beta\\ \beta \\ \alpha\end{pmatrix}$ →←.

(給分標準): 只有答案，沒有解釋，扣2分. 只有答案，解釋不正確，也扣2分.

(c) Let $y_0=\begin{pmatrix}\alpha+\beta\\ \alpha \\ \beta\end{pmatrix}$ since $y_0 \in R(A)=\left\{\begin{pmatrix}\alpha+\beta\\ \beta \\ \alpha\end{pmatrix} \mid \alpha,\beta \in \mathbb{R}\right\}$.

By hypothesis, we have $\begin{cases} \langle b-y_0, \begin{pmatrix}1\\0\\1\end{pmatrix}\rangle = 0 \\ \langle b-y_0, \begin{pmatrix}1\\1\\0\end{pmatrix}\rangle = 0 \end{cases}$

$\Rightarrow \alpha=\frac{2}{3}$, $\beta=\frac{2}{3}$.

So, $y_0=\begin{pmatrix}\frac{4}{3}\\ \frac{2}{3} \\ \frac{2}{3}\end{pmatrix}$.

(給分標準): ① 計算錯誤，扣5分. ② 只有答案，不予計分.

(4) (25%)

$$A=\begin{pmatrix}2 & 0 & \sqrt{2}\\ 0 & 1 & 0\\ \sqrt{2} & 0 & 3\end{pmatrix}$$

10 % (a) Find the eigenvalues and eigenvectors of A.

5 % (b) Find a matrix Q such that $Q^{-1}AQ$ is diagonal.

5 % (c) Find a matrix B such that $B^2=A$.

5% (d) Compute A^{10}.

(a) Consider $\det(\lambda I - A) = 0$, i.e., $\begin{vmatrix} \lambda-2 & 0 & -\sqrt{2} \\ 0 & \lambda-1 & 0 \\ -\sqrt{2} & 0 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2(\lambda-4) = 0$.

So, the eigenvaluesof A : $1, (1), 4$.

For $\lambda = 1$:

Conoider the eigenspace of A corresponding to $\lambda=1$, $N(I-A) = \{x \mid (I-A)(x) = 0\}$.

$\Rightarrow \begin{pmatrix} -1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x = \begin{pmatrix} t \\ s \\ \sqrt{2}t \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t, s \in \mathbb{R}$.

For $\lambda = 4$:

Conoider the eigenspace of A corresponding to $\lambda=4$. $N(4I-A) = \{x \mid (4I-A)(x) = 0\}$.

$\Rightarrow \begin{pmatrix} 2 & 0 & -\sqrt{2} \\ 0 & 3 & 0 \\ -\sqrt{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x = \begin{pmatrix} \alpha \\ 0 \\ \sqrt{2}\alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$.

Therefore, the conclusion is that (i) the eigenvalues : $1, (1), 4$.

(ii) eigenvectors : $\begin{pmatrix} t \\ s \\ \sqrt{2}t \end{pmatrix}$ corresponding to $\lambda = 1$

$\begin{pmatrix} \alpha \\ 0 \\ \sqrt{2}\alpha \end{pmatrix}$ " $\lambda = 4$

(給分標準): ① 沒有過程，一律不計分

② 有 eigenvalues, 無 eigenvectors, 扣 5 分.

③ 其它方面, 酌量給分.

(b)
$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \end{pmatrix}$$

(給分標準): ① 由(a)決定，若(a)部分的 eigenvectors 有錯，但此處 By (a) 來定，那麼不予扣分.

② 只有答案，且(a)不正確，依然不予計分.
(正確)

(c)

By (a) and (b), We have

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = Q A Q^{-1}$$

$$\Rightarrow \quad Q^{-1} D Q = A = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q$$

Let $B = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q$, then $B^2 = A$.

(給分標準): 撇開 Q 不談，只要有 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 出現，且 $B = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q$，則給 5 分. 若無，酌量計分.

(d)
By (c), $A = Q^{-1} D Q \Rightarrow A^{10} = Q^{-1} D^{10} Q$

(給分標準): ① 由(c)決定，一定要有 $A^{10} = Q^{-1} D^{10} Q$ 的型式出現. (給 5 分).

② 直接來算者，也給 5 分.

(5) (20 pts).

5% (a) If A is diagonalizable and B is similar to A, then B is diagonalizable.

5% (b) If A has distinct eigenvalues, then A is diagonalizable.

5% (c) If the eigenvalues of A are not distinct, then A is not diagonalizable.

5% (d) If A and B are both diagonalizable with the same diagonalizing matrix, then $AB=BA$.

Sol:

(a) True.

Since $A \sim D$ and $A \sim B$, we have $B \sim D$.

(b) True. Let A be an $n\times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$, where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Choose $x_i \in N(\lambda_i I - A)$; then claim $\{x_1, \cdots, x_n\}$ is linearly independent.

Use $\{x_1, \cdots, x_n\}$ is linearly independent, then we have:

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = Q^{-1} A Q, \text{ where } Q = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

Pf of claim: Consider $a_1x_1 + \cdots + a_nx_n = 0$. If there existed $a_i \neq 0$ for some i. Without loss of generality, we may assume $a_1 \neq 0$. Then we have:

$$-x_1 = a_1^{-1}a_2 x_2 + \cdots + a_1^{-1}a_n x_n.$$

$$\Rightarrow A(-x_1) = a_1^{-1}a_2 A(x_2) + \cdots + a_1^{-1}a_n A(x_n)$$

$$\Rightarrow -\lambda_1 x_1 = a_1^{-1}a_2 \lambda_2 x_2 + \cdots + a_1^{-1}a_n \lambda_n x_n$$

$$\Rightarrow \lambda_1 a_1^{-1}a_2 x_2 + \cdots + \lambda_1 a_1^{-1}a_n x_n = a_1^{-1}a_2 \lambda_2 x_2 + \cdots + a_1^{-1}a_n \lambda_n x_n$$

$$\Rightarrow a_1^{-1}a_2(\lambda_1 - \lambda_2)x_2 + \cdots + a_1^{-1}a_n(\lambda_1 - \lambda_n)x_n = 0$$

$$\Rightarrow a_2(\lambda_1 - \lambda_2)x_2 + \cdots + a_n(\lambda_1 - \lambda_n)x_n = 0.$$

$\vdots$ Reduced

$\rightarrow\leftarrow$ ($\lambda_i \neq \lambda_j$ if $i \neq j$).

(c) False.

example: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, etc.

or I_n $C_{n\times 4}$.

(d) True. Say $D_1 = X A X^{-1}$, where D_1 is a diagonal matrix

$D_2 = X B X^{-1}$, " D_2 "

$$\Rightarrow \quad AB = X^{-1} D_1 X \, X^{-1} D_2 X = X^{-1} D_1 D_2 X$$

$$BA = X^{-1} D_2 X \, X^{-1} D_1 X = X^{-1} D_2 D_1 X$$

Note that $D_1 D_2 = D_2 D_1$

給分標準: ① 只有答案，沒有過程，每小題一律扣2分.

② 全對給5分.