MATH 543
METHODS OF APPLIED MATHEMATICS I
Homework Set III

For November 19, 2009

QUESTIONS

1. Prove that each of the following sequences define the Dirac δ-function: (For all cases $n = 1, 2, \cdots$)
(i) $D_n^1 = \frac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$
(ii) $D_n^2 = \frac{1-\cos nx}{\pi n x^2}$
(iii) $D_n^3 = \frac{n}{\pi} \frac{1}{1+n^2x^2}$
(iv) $D_n^4 = \frac{\sin nx}{\pi x}$. 1.
2. Higher dimensional delta functions can be defined. For instance in three dimensions the delta function $\delta^3(\mathbf{x})$ can be represented in terms one dimensional delta functions. The way we achieve this is to use the general identity $\int_V \delta^3(\mathbf{x}) d^3x = 1$. Hence using this property prove that
(a) In Cartesian coordinates $\delta^3(\mathbf{x}) = \delta(x)\,\delta(y)\,\delta(z)$
(b) In spherical coordinates $\delta^3(\mathbf{x}) = \frac{1}{r^2 \sin\theta}\,\delta(r)\,\delta(\theta)\,\delta(\varphi)$
(c) In cylindrical coordinates $\delta^3(\mathbf{x}) = \frac{1}{\rho}\,\delta(\rho)\,\delta(z)\,\delta(\varphi)$
(d) In three dimensions verify the following identity and find the constant α_3 in the following equation

$$\nabla^2 \frac{1}{|\mathbf{x}|} = \alpha_3\, \delta^3(\mathbf{x})$$

where ∇^2 is the three dimensional Laplace operator (Laplacian).
(e) The above Laplace equation can be written for in any dimension n

$$\nabla^2 \frac{1}{|\mathbf{x}|^{n-2}} = \alpha_n\, \delta^n(\mathbf{x})$$

where $\delta^n(\mathbf{x})$ is the n dimensional δ function and α_n are just constants. Here $n \neq 2$. For $n = 2$ we have

$$\nabla^2 \log|\mathbf{x}| = \alpha_2\, \delta^2(\mathbf{x})$$

Find α_n, $n \neq 2$ and α_2

3. Find the Green's function for $L_x = \frac{d^2}{dx^2} + k^2$ with $u(0) = u(a) = 0$ and solve the boundary value problem $l\,u = f$, where f is a given function.

4. Solve $u'' = f(x)$ with $u(0) = 0$ and $u'(1) = 1$

5. Let

$$\frac{1}{w}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + cu = f, \quad a < x < b,$$

and the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$ are defined at $x = a$ and $x = b$ respectively. Solve the Green's function give the solution of the problem. Discuss all possibilities.