MATH 543
METHODS OF APPLIED MATHEMATICS I
First Homework Set

For September 28, 2009

QUESTIONS

1. In an inner product space, we define the norm of an element x of a vector space X to be $||x|| = \sqrt{< x, x >}$ (we use the notation of the lecture notes). Prove the following

a. $||x|| > 0$ if $x \neq 0$.
b. $||\alpha x|| = |\alpha|||x||$.
c. $| < x, y > | \leq ||x||||y||$.
d. $||x + y|| \leq ||x| + ||y||$.
e. $||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$.
f. If $< x, y >= 0$, then $||x + y||^2 = ||x||^2 + ||y||^2$.

Remark: Inner Product Spaces. Let X be a linear vector space over the $\mathbb{C}$. Let $x, y, z \in X$ and α. Then an inner product space X over the complex numbers with an inner product $<, >$ satisfy the following conditions

a. $< x, y >$ is a complex number.
b. $< x, y >= \overline{< y, x >}$.
c. $< \alpha x, y >= \alpha < x, y >$.
d. $< x, x >> 0$ if $x \neq 0$.
e. $< x + y, z >=< x, z > + < y, z >$.

2. Show that the space $C[a, b]$ with sup norm is a Banach space (a complete normed space).

3. Show that the space $C[a, b]$ with L^2 norm is not Banach space (a complete normed space). Prove this by counter example. Let the interval be $I = [-1, 1]$ and define a sequence $f_k(x)$ as

$f_k(x) = 1, \quad \text{for } \frac{1}{k} < x \leq 1,$
$f_k(x) = \frac{kx+1}{2}, \quad \text{for } \frac{-1}{k} < x \leq \frac{1}{k}$
$f_k(x) = 0, \quad \text{for } -1 \leq x < -\frac{1}{k}$

Prove that this is a Cauchy sequence but does not converge to continuous function.