SET 9

MATH 543: PERTURBATION

Reference: David Logan.

About the uniform convergence of a perturbation series we have mainly the following three definitions

Definition 1: *Let $f(t,\varepsilon)$ and $g(t,\varepsilon)$ be defined for all $t \in I$ in some neighborhood (or punctured neighborhood) of $\varepsilon = 0$. We write*

$$f(t,\varepsilon) = o(g(t,\varepsilon)) \ \text{as} \ \varepsilon \to 0$$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(t,\varepsilon)}{g(t,\varepsilon)} \right| = 0$$

pointwise on I. If the limit above is uniform on I, we write $f(t,\varepsilon) = o(g(t,\varepsilon))$ uniformly on I. If there exists a positive function $M(t)$ on I such that

$$|f(t,\varepsilon)| \leq M(t)|g(t,\varepsilon)|$$

for all $t \in I$ and ε in some neighborhood of zero, then we write

$$f(t,\varepsilon) = O(g(t,\varepsilon)) \ \text{as} \ \varepsilon \to 0, \ uniformly \ on \ I$$

Definition 2. *A function $y_a(t,\varepsilon)$ is a uniformly valid approximation to a function $y(t,\varepsilon)$ on an interval I as $\varepsilon \to 0$ if the error function $E(t,\varepsilon)$ defined by*

$$E(t,\varepsilon) = y(t,\varepsilon) - y_a(t,\varepsilon)$$

converges to zero as $\varepsilon \to 0$ uniformly for all $t \in I$

Definition 3. *Let* $F(t, y(t), y'(t), \cdots, \varepsilon)$ *be a differntial equation. We say that an approximate solution* $y_a(t, \varepsilon)$ *satisfies the differential equation uniformly for* $t \in I$ *as* $\varepsilon \to 0$ *if the residual error function* $r(t, \varepsilon)$ *goes to zero*

$$r(t, \varepsilon) = F(t, y_a(t, \varepsilon), y_a'(t, \varepsilon), \cdots, \varepsilon) \to 0$$

uniformly on I.

Problems:
(1) Let $y' + y - \varepsilon y^2 = 0$ for $t > 0$ with $y(0) = 1$. Find approximate solutions (up to second order perturbation for instance) and discuss whether the approximate solution is a uniformly valid solution.
(2) Let $y'' + y + \varepsilon y^3 = 0$ for $t > 0$ with $y(0) = 1, \quad y'(0) = 0$. Solve this equation up to second order in ε.
(3) Let $y'' + \varepsilon(y')^2 + y = 0$ for $t > 0$ with $y(0) = 1, \quad y'(0) = 0$. Find a first order approximation of this problem and discuss the validity of the approximation.
(4) Let $y'' - y = \varepsilon ty$ for $t > 0$ with $y(0) = 1, \quad y'(0) = -1$. Find a first order approximation of the boundary value problem and discuss the validity of the approximation.
(5) Use the Poincare'-Lindstedt method to obtain a two term (first order perturbation) perturbation approximation to the following problems:
(i) $y'' + y = \varepsilon y(y')^2, \; y(0) = 1, \; y'(0) = 0$.
(ii) $y'' + 9y = 3\varepsilon y^3, \; y(0) = 0, \; y'(0) = 1$.
(*iii*) $y'' + y = \varepsilon y(1 - y'^{\,2}, \; y(0) = 1, \; y'(0) = 0$
(6) Find a two term perturbation solution of

$$y' + y = \frac{1}{1 + \varepsilon y},$$

for $t > 0$ with $y(0) = 0$ and $\varepsilon << 1$. Compare with the exact solution and discuss the validity of the approximation.

SINGULAR PERTURBATIONS: BOUNDARY LAYER ANALYSIS:

Theorem: *Consider the boundary value problem*

$$\begin{aligned} \varepsilon y'' + p(t)y' + q(t)y &= 0, 0 < t < 1, 0 < \varepsilon << 1, \\ y(0) &= a, \quad y(1) = b \end{aligned}$$

where p and q are continuous functions on $0 \leq t \leq 1$ and $p(t) > 0$ for $0 \leq t \leq 1$. Then there exists a boundary layer at $t = 0$ with inner and outer approximations given by

$$y_i(t) = C_1 + (a - C_1)e^{-p(0)\,t/\varepsilon},$$
$$y_o(t) = b\,exp\left(\int_t^1 \frac{q(s)}{p(s)} ds\right)$$

where

$$C_1 = bexp\left(\int_0^1 \frac{q(s)}{p(s)} ds\right)$$

Problems:

(7) Consider the following boundary value problem

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y &= 0, \ 0 < t < 1, \ 0 < \varepsilon << 1, \\ y(0) &= 0, \ y(1) = 1 \end{aligned}$$

Use singular perturbation methods to obtain a uniform approximate solution.

(8) Consider the boundary value problems

$$
\begin{aligned}
(i) \qquad \varepsilon y'' + y' &= 2t, \ 0 < t < 1, \ 0 < \varepsilon << 1, \\
y(0) &= 1, \ y(1) = 1, \\
(ii) \ \ \varepsilon y'' + y' + y &= 0, \ t > 0, \\
y(0) &= 0, \ \varepsilon \, y'(0) = 1.
\end{aligned}
$$

Use the singular perturbation methods to obtain a uniform approximate solution.

(9) Consider the following boundary value problem

$$
\begin{aligned}
y' &= -y + (y + \lambda)z, \\
\varepsilon z' &= y - (y + \mu)z, \\
y(0) &= 1, \quad z(0) = 0,
\end{aligned}
$$

Use singular perturbation methods to obtain a uniform approximate solution of this system of first order boundary value problem.

(10) Use singular perturbation methods to obtain a uniform approximate solution to the following problems. In each case assume $0 < \varepsilon << 1$ and $0 < t < 1$.

(i) $\varepsilon y'' + 2y' + y = 0, \quad y(0) = 0, \ y(1) = 1,$
(ii) $\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4, \ y(1) = 1/2,$
(iii) $\varepsilon y'' + (1 + t)y' = 1, \quad y(0) = 0, \ y(1) = 1 + \ln 2,$
(iv) $\varepsilon y'' + (t + 1)y' + y = 0, \quad y(0) = 0, \ y(1) = 1,$
(v) $\varepsilon y'' + t^{1/3} y' + y = 0, \quad y(0) = 0, \ y(1) = exp(-3/2),$
(vi) $\varepsilon y'' + ty' - ty = 0, \quad y(0) = 0, \ y(1) = e,$
(vii) $\varepsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0,$
(viii) $\varepsilon y'' - (2 - t^2)y = -1, \quad y'(0) = 0, \ y(1) = 1.$

THE WKB APPROXIMATION

The WKB approximation is a perturbation method that we shall apply to the follwong type of boundary value problems

$$\varepsilon^2 y'' + q(x)y = 0, \quad 0 < \varepsilon << 1,$$
$$y'' + (\lambda^2 p(x) + q(x))y = 0, \quad \lambda >> 1,$$
$$y'' + q(\varepsilon\, x)^2\, y = 0, \quad 0 < \varepsilon << 1$$

We sall here consider only the first type of equations. Depneding on the function $q(x)$ we have two different types

Nonoscillatiry Case: $q(x) = -k(x)^2$ where $k(x) > 0$ then the WKB solution takes the form

$$y_{WKB} = \frac{C_1}{\sqrt{k(x)}}\, exp\left(\frac{1}{\varepsilon}\int_a^x k(\xi)d\xi\right) + \frac{C_2}{\sqrt{k(x)}}\, exp\left(-\frac{1}{\varepsilon}\int_a^x k(\xi)d\xi\right) \quad (1)$$

Oscillatory Case: $q(x) = k(x)^2$ where $k(x) > 0$ then the WKB solution takes the form

$$y_{WKB} = \frac{C_1}{\sqrt{k(x)}}\, exp\left(\frac{i}{\varepsilon}\int_a^x k(\xi)d\xi\right) + \frac{C_2}{\sqrt{k(x)}}\, exp\left(-\frac{i}{\varepsilon}\int_a^x k(\xi)d\xi\right) \quad (2)$$

To find the arbitrary constants C_1 and C_2 we need the boundary conditions. As an illustration consider the following example. Find the WKB solution and eigenvalues λ corresponding to these solutions

$$y'' + \lambda q(x)y = 0,\ 0 < x < \pi, \quad y(0) = y(\pi) = 0,$$

where $q(x) > 0$ for all $x \in [0, \pi]$. This is oscillatory case with $C_2 = 0$

$$\lambda = \lambda_n = \frac{n^2\pi^2}{\left(\int_0^\pi \sqrt{q(\xi)}d\xi\right)^2}$$

for large integer n.

Problems: (11) Find the WKB approximation of the following problem

$$y'' - \lambda(1+x^2)\,y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

for large λ.

(12) Show that the large eigenvalues of the problem

$$y'' + \lambda(x+\pi)^4\,y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

are given by

$$\lambda = \lambda_n = \frac{9n^2}{49\pi^4}$$

for large integers n , and find the corresponding eigenfunctions

(13) The slowly varying oscillator equation is

$$y'' + q(\varepsilon x)^2 y = 0$$

where $\varepsilon << 1$ and q is a strictly positive function. Find and approximate solution. Why is the equation called *slowly varying*?

(14) Find an approximation for the large eigenvalues of

$$y'' + \lambda e^{4x}\,y = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0.$$

(15) Find an approximation for the general solution of

$$y'' + (\lambda^2 x^2 + x)y = 0, \quad x > 0, \quad \lambda >> 1$$

proceed as in the above oscillatory case by letting

$$y = e^{\frac{iu}{\varepsilon}}, \quad \varepsilon = \frac{1}{\lambda}$$

ASYMPTOTIC EXPANSION OF INTEGRALS

Laplace Integrals

Theorem 2: *Consider the integral*

$$I(\lambda) = \int_0^b t^\alpha \, h(t) \, e^{-\lambda t} \, dt, \tag{3}$$

where $\alpha > -1$, $h(t)$ *has a Taylor series expansion in the neighborhood* $t = 0$, *with* $h(0) ne 0$ *and where* $|h(t)| \leq k e^{ct}$ *for some positive constants* k *and* c. *Then (approximately)*

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{h^n(0)\Gamma(\alpha+n+1)}{n!\lambda^{\alpha+n+1}} + EST, \quad \lambda \to \infty \tag{4}$$

where EST *means exponentially small terms.*

This theorem is sometimes called as the Watson's Lemma.

Problems:

(16) Prove that

$$I(\lambda) = \int_0^\infty \frac{\sin t}{t} e^{-\lambda t} \, dt,$$
$$= \frac{1}{\lambda} - \frac{2!}{3!\lambda^3} + O(\frac{1}{\lambda^5}) + EST$$

for large λ ($\lambda >> 1$).

(17) Prove that

$$\frac{2}{\sqrt{2}} \int_\lambda^\infty e^{-t^2} \, dt \sim \frac{2}{\sqrt{2}} e^{-\lambda^2} \left(\frac{1}{2\lambda} - \frac{\Gamma(3)}{(2\lambda)^3} + \frac{\Gamma(5)}{2!(2\lambda)^5} - \cdots \right) \tag{5}$$

Lemma 3: *Let*

$$I(\lambda) = \int_a^b f(t) e^{\lambda g(t)} \, dt$$

where f is continuous and g is sufficiently smooth and has a unuque maximum at a point $x = c$ in (a, b). Then

$$I(\lambda) \sim f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}, \quad \lambda >> 1$$

Remark: If the point $x = c$ happens to be at one of the end points a or b then the right hand side of the above expression should be divided by 2.

Lemma 3: *Let*

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$$

where f is continuous and the maximum of g occurs at $t = b$ one of the end point of the interval with $g'(t) > 0$ for all $t \in [a, b]$. Then

$$I(\lambda) \sim \frac{f(b)\, e^{\lambda g(b)}}{\lambda g'(b)}$$

for $\lambda >> 1$.

Problems : (18) Verify the following approximations for large λ.

(i) $\int_0^\infty e^{-\lambda t} \ln(1 + t^2) dt \sim \frac{2!}{\lambda^3} - \frac{1}{2}\frac{4!}{\lambda^5} + \cdots, \quad \lambda >> 1,$
(ii) $\int_0^1 \sqrt{1 + t}\, e^{\lambda(2t - t^2)}\, dt \sim \sqrt{\frac{\pi}{2\lambda}}\, e^{\lambda}. \quad \lambda >> 1,$
(iii) $\int_1^2 \sqrt{3 + t}\, e^{\frac{\lambda}{t+1}} dt \sim \frac{8}{\lambda} e^{\lambda/2}, \quad \lambda >> 1$

(19) Find a two term approximation for large λ for

$$\int_0^{\pi/2} e^{-\lambda \tan^2\theta}\, d\theta$$