SET 8

MATH 543: INTEGRAL REPERSENTATIONS

References: DK and Hildebrandt

Let $L_z\, u(z) = 0$ be a homogenous linear differential equation. Let the integral representation of the solution of this differential equation be given by

$$u(z) = \int_I K(z,t)\, v(t)\, dt, \tag{1}$$

where $K(z,t)$ is the kernel , $v(t),\; t \in I$ is a continuous function and I is an interval to be determined. To determine these three quantities we follow the steps (i) Assume that there exists an operator such that

$$L_z\, K(z,t) = M_t\, K(z,t) \tag{2}$$

Hence the first step is to determine the operator M_t if the *Kernel* of the integral representation is known. (ii) If (2) holds then

$$\begin{aligned} L_z u(z) &= \int_I L_z\, K(z,t)\, v(t)\, dt, && (3)\\ &= \int_I [M_t\, K(z,t)]\, v(t)\, dt && (4) \end{aligned}$$

Using the generalized Green's identity we obtain

$$L_z\, u(z) = \int_I K(z,t)\, M_t^{\dagger}\, v(t)\, dt + Q(K,v)|_I = 0 \tag{5}$$

where $M_t^{\dagger}$ is adjoint of M_t and Q is the surface term resulting from the identity

$$v(t) M_t K(z,t) - K(z,t) M_t^{\dagger} v(t) = \frac{dQ(K,v)}{dt}, \tag{6}$$

Then we let

$$M_t^\dagger v(t) = 0, \text{and} \quad Q(K,v)|_I = 0, \tag{7}$$

(iii) third step is to find $v(t)$ from the first equation in (7) and (iv) fourth step is to determine I from the second equation in (7).

The choice of Kernels:
(1) *Laplace kernel*. $K(z,t) = e^{-zt}$. It is used when the differential equation $Lu = 0$ has coefficients linear in z
(2) *Euler kernel*: $K(z,t) = (z-t)^\nu$. It is used when the coefficient of nth order derivative in $Lu = 0$ is a polynomial of degree n. Here ν is a constant to be determined.
(3) *Bessel kernel*: $K(z,t) = (z/2)^\nu \, e^{t - z^2/(4t)}$ where ν is a constant to be determined.

For other types of kernels see (For instance *Fourier*, *Melin*, *Hankel*) Sadri Hassan Chapter 15. As a summary We first decide on the kernel and use the above four steps.

Problems
1. Hypergeometric function.

$$L_z u(z) = [z(1-z)\,\frac{d^2}{dz^2} + [c - (a+b+1)\frac{d}{dz} - ab]\,u(z) = 0 \tag{8}$$

Choose the Euler kernel $K(z,t) = (z-t)^\nu$ and prove that $\nu = -a$ (or $-b$) and (for the choice $\nu = -a$)
(a)

$$M_t = (t^2 - t)\,\frac{d^2}{dt^2} - [(b-a-1)t + a - c + 1]\frac{d}{dt}$$

(b)

$$v(t) = kt^{a-c}\,(t-1)^{c-b-1}, \quad k \text{ is a constant}$$

(c)

$$Q(K,v) = akt^{a-c+1}(t-1)^{c-b}(z-t)^{-a-1}$$

(d) There are several choices of the interval I. One of them is $I = [1, \infty)$. Hence

(e)

$$u(z) = k \int_1^\infty (t-z)^{-a} t^{-a-c} (t-1)^{c-b-1} dt$$

Provided $Rec > Reb > 0$. One can find the constant k by letting first $a = 0$ and then $z = 0$ the last equation becomes

$$\begin{aligned} u(0) &= 1 = k \int_1^\infty t^{-a-c} (t-1)^{c-b-1} dt \\ &= k \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \end{aligned} \tag{9}$$

then we find an integral representation of the hypergeometric function which is defined not only in the disc $|z| < 1$ but in a more larger region $0 \leq arg(1-z) \leq 2\pi$ (prove this claim).

$$u(z) = F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty (t-z)^{-a} \, t^{a-c} \, (t-1)^{a-c-1} \, dt$$

By making $t \to 1/t$ transformation one finds also that

$$u(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-tz)^{-a} \, t^{b-1} \, (1-t)^{c-b-1} \, dt$$

2. Confluent hypergeometric equation:

$$L_z u(z) = [z \frac{d^2}{dz^2} + (c-z) \frac{d}{dz} - a] u(z) = 0 \tag{10}$$

Choose the Laplace kernel $K(z, t) = e^{zt}$ and prove that

(a)

$$M_t = (t^2 - t) \frac{d^2}{dt^2} + ct - a$$

Hence

(b)

$$v(t) = k t^{a-1} \, (1-t)^{c-a-1}, \quad Q(K, v) = -k e^{zt} t^a (1-t)^{c-a}$$

One can choose $I = [0, 1]$ provided $Rec > Rea > 0$.
(c)

$$u(z) = \Phi(a, c; z) = \Gamma(c)\Gamma(c-a)\Gamma(a) \int_0^1 e^{zt} t^{a-1}(1-t)^{c-a-1} dt$$

3. Find the above integral representation from the above integral representation of the hypergeometric function in Problem 1 by using the formula

$$\Phi(a, c; z) = \lim_{b \to \infty} F(a, b; c; z/b)$$

4. Bessel's Equation:

$$L_z u(z) = [\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 - \frac{\nu^2}{z^2}]u = 0 \qquad (11)$$

Choose the kernel $K(z, t) = (z/2)^\nu e^{t - t^2/4t}$ and show that
(a)

$$M_t = \frac{d}{dt} - \frac{\nu + 1}{t}$$

and
(b)

$$v(t) = k t^{-\nu-1}, \quad Q(K, v) = k(z/2)^\nu e^{t - z^2/4t}$$

Then one finds and integral representation of the Bessel function $J_\nu(z)$ given by

$$J_\nu(z) = k(z/2)^\nu \int_C t^{-\nu-1} \, e^{t - z^2/4t} \, dt$$

where k is a constant to be determined and C is a closed contour closing the axis $Rez \leq 0$. We find k by letting first $t = \frac{uz}{2}$ which leads to

$$J_\nu(z) = k \int_C u^{-\nu-1} \, e^{z(u-1/u)/2} du$$

and letting $z = 0$ in the above equation with $\nu = 0$ then

$$J_0(0) = 1 = k \int_C \frac{dt}{t}$$

Hence $k = 2\pi i$.

5. Hermite equation:

$$[\frac{d^2}{dz^2} - z\frac{d}{dz} + n]u(z) = 0 \tag{12}$$

where n is a positive integer. Prove that the Hermite function $H_n(z)$ has an integral representation

$$H_n(z) = \frac{1}{2\pi i \Gamma(n+1)} \int_C t^{-n-1} s^{zt}\, e^{-t^2/2} dt$$

where C is closed contour closing the origin. Consider also the transformation $s = t - z$ and find the Rodriquez formula for the Hermite polynomials.