SET 7

MATH 543: FUCHSIAN DIFFERENTIAL EQUATIONS HYPERGEOMETRIC FUNCTION

References: DK and Sadri Hassan.

Historical Notes: Please read the book *Linear Differential Equations and the Group Theory* by *Jeremy J. Gray* , *Birkhouser, 2000* for the contributions of **Euler, Pfaff , Gauss, Riemann, Kummer, Jacobi** and others on the P symboll and on the Hypergeometric equation and hyoergeometric function.

Definition 1. Linear ordinary differential equations having only regular singular points are called *Fuchsian Differential Equations* (FDE).

FDE with two regular singular points

Definition 2. (Regular singular point at ∞). If the transformed DE by $z = \frac{1}{t}$ has regular singular point at $t = 0$ then the original DE has a regular singular point (in the extended complex plane) at $z = \infty$.

Proposition 1. *Second order FDE having only two singular points is equivalent to a DE with constant coefficients, hence solvable in terms of the elementary functions* $\sin z$,$\cos z$ *and polynomial in z.*

FDE with three regular singular points: The Riemann equation and Riemann P Symbol.

Riemann has put a second order linear FDE with three regular singular points $z = z_1$, $z = z_2$ and $z = z_3$ into the form

$$u'' + \left(\frac{1+\alpha+\alpha'}{z-z_1}\frac{1+\beta+\beta'}{z-z_2} + \frac{1+\gamma+\gamma'}{z-z_3}\right) + \left(\frac{(z_1-z_2)(z_1-z_3)\alpha\,\alpha'}{z-z_1} + \right.$$
$$\left.\frac{(z_2-z_1)(z_2-z_3)\beta\beta'}{z-z_2} + \frac{(z_3-z_1)(z_3-z_1)\gamma\,\gamma'}{z-z_3}\right)\frac{u}{(z-z_1)(z-z_2)(z-z_3)} = 0, \quad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are constant satisfying the constraint

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

The solution of (1) is denoted as the Riemann P-symbol

$$u(z) = P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} \tag{2}$$

The columns in the P symbol indicate the location of the regular singular points and the corresponding roots of the indical equation, i.e.,

1. $z = z_1$, $r_1 = \alpha$ and $r_2 = \alpha'$.
2. $z = z_2$, $r_1 = \beta$ and $r_2 = \beta'$.
3. $z = z_3$, $r_1 = \gamma$ and $r_2 = \gamma'$.

One can reduce the 9 number of parameters in the Riemann equation (or in the Riemann P symbol) into three parameters by using the following type of transformation

(i) $u(z) = (z - z_1)^r \, (z - z_2)^s \, (z - z_3)^t \, v(z), \quad r + s + t = 0$
(ii) The Mobius transformation: $z' = \frac{(z-z_1)\,(z_3-z_2)}{(z-z_2)\,(z_3-z_1)}$

we obtain first (from (i) above)

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2} \right)^{\alpha} \left(\frac{z - z_3}{z - z_2} \right)^{\gamma} P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ 0 & a & 0 & z \\ 1 - c & b & c - a - b & \end{array} \right\} \tag{3}$$

and the using (ii) we obtain

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2} \right)^{\alpha} \left(\frac{z - z_3}{z - z_2} \right)^{\gamma} P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1 - c & b & c - a - b & \end{array} \right\}, \tag{4}$$

where

$$z' = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}, \quad a = \alpha+\beta+\gamma, \tag{5}$$

$$b = \alpha+\beta'+\gamma, \quad c = 1+\alpha-\alpha' \tag{6}$$

The P symboll in the right hand side of (4) is the hypergeomtric function, $F(a,b;c;z')$

$$F(a,b;c;z) = P\left\{\begin{array}{llll} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{array}\right\} \tag{7}$$

Problems

1. Prove that the hypergeometric function defined above (7) satisfies the differential equation (hypergeometric DE)

$$z(1-z)\,u'' + [c-(1+a+b)\,z)\,u' - ab\,u = 0 \tag{8}$$

2. Find (Kummer) transformations leaving the Reiemann equation form invariant.

3. Prove that

$$F = \sum_{n=0}^{\infty} \frac{a_n\, b_n}{c_n} \frac{z^n}{n!}$$

where

$$a_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad b_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad c_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad |z| \leq 1$$

4. Prove the following Proposition.

Proposition 2. *Solutions of the hypergeometric function bout its regular singular points $z=0$, $z=\infty$ and $z=1$, provided $1-c$, $b-a$, and $c-a-b$ are not integers, are respectively given by*

$$
\begin{aligned}
u(z) &= A_1\, F(a,b;c;z) + B_1\, z^{1-c}\, F(b-c+1, a-c+1; 2-c; z), && (9)\\
u(z) &= A_2\, z^{-a}\, F(a, a-c+1; a-b+1; \frac{1}{z}) + B_2\, z^{-b} \\
&\quad F(b, b-c+1; b-a+1; \frac{1}{z}), && (10)\\
u(z) &= A_3\, F(a,b; a+b+1-c; 1-z) + B_3\, z^{c-a-b}\, F(c-b, c-a; \\
&\quad 1+c-a-b; 1-z), && (11)
\end{aligned}
$$

where $A_i, B_i, (i = 1,2,3)$ *are arbitrary constants.*

5. Find the solutions (10) and (11) about the regular singular points $z = \infty$ and $z = 1$ by the use of *Kummer's* transformations mentioned in the Pr.2.

6. Prove the following: The Jacobi function $P_\lambda^{(\alpha,\beta)}$ satisfying the Jacobi equation equation

$$(1-z^2)u'' + [\beta - \alpha - (\alpha+\beta+2)z]u' + \lambda(\lambda+\alpha+\beta+1)u = 0$$

can be written in terms of the hypergeometric function

$$P_\lambda^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\, F(-\lambda, \lambda+\alpha+\beta; \alpha+1; \frac{1-z}{2})$$

When $\lambda = n$ a non-negative integer the the solution becomes Jacobi polynomials containing the Legendre, and Tchbechev polynomials.

7. Prove that when the regular singular points $z = z_2$ and z_3 of the Riemann equation are pushed out ∞ then the resulting function function is the confluent hypergeometric function $\Phi(a, c; z)$ in

8. Prove that $\Phi(a, c, z) = \lim_{b\to\infty} F(a, b; c; \frac{z}{b})$

9. Prove that the confluent hypergeometric function satisfies the DE

$$z\, u'' + (c - z)\, u' - a\, u = 0 \qquad (12)$$

10. Prove that the point $z = \infty$ of the confluent hypergeometric equation (12) is an irregular singular point.

11. Find solutions of the confluent hypergeometric equation about all it regular singular points. The form of the equation is of the form

$$u(z) = A\,\Phi(a, c; z) + B\,z^{1-c}\,\Phi(a', c', b'\,z)$$

where A and B are arbitrary constants. Find the constants a', c' and b'

12. Prove that Bessel's function $J_v(z)$ satisfying the Bessel equation $u'' + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0$ is given by

$$J_v(z) = \frac{1}{\Gamma(\nu+1)}(z/2)^\nu\,\Phi(\nu + 1/2, 2\nu + 1; 2iz)$$

13. Prove that

$$\frac{d^n}{dz^n}\,F(a, b; , c; z) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)}\,F(a+n, b+n; c+n; z)$$

Use induction.

14. Prove the following:

(i) $F(-a, b; b; -z) = (1+z)^a$,

(ii) $F(1, 1; 2; -z) = \frac{1}{z}\ln(1+z)$,

(iii) $F(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2) = \frac{1}{z}\sin^{-1}\,z$,

(iv)

$$F(\frac{1}{2}, \frac{1}{2}; 1; z^2) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - z^2\sin^2\theta}},$$

(v) $e^z = \lim_{b\to\infty} F((1, b, 1, \frac{z}{b})$

(vi) The error function

$$Erf(z) = \int_0^\infty e^{-t^2}\,dt$$

15. Prove that the Hermite-Weber differential equation

$$u'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)u = 0$$

can be converted to the confluent hypergeometric equation by

$$u(z) = e^{\frac{-z^2}{4}}\,v(\xi), \quad \xi = \frac{z^2}{2}$$

with

$$v(\xi) = \Phi(-\nu/2, 1/2; \xi)$$