SET 5

MATH 543: GREEN'S FUNCTIONS

(References: DK, Sadri Hassan and Hildebrandt)

Generalized Green's Identity

Let L_x be a differential operator and u , v are some functions in $L^2_w(a,b)$ then the *adjoint operator* $L^\dagger$ is defined through the *Lagrange identity*

$$w[\bar{v}L_x u - u\,\overline{\{L_x^\dagger v\}}] = \frac{d}{dx}Q(u,\bar{v}), \tag{1}$$

where $w > 0$ is the weight function and $Q(u,\bar{v})$ has in general the following form

$$Q(u,\bar{v}) = Au\bar{v} + B\bar{v}\frac{du}{dx} + Cu\frac{d\bar{v}}{dx} + D\frac{du}{dx}\frac{d\bar{v}}{dx}, \tag{2}$$

Integrating the above identity over the interval $[a,b]$ we get the *generalized Green's identity*

$$\int_a^b w(x)\,[\bar{v}\,L_x u - u\,\overline{\{L_x^\dagger\,v\}}]dx = Q(u(b),\bar{v}(b)) - Q(u(a),\bar{v}(a)), \tag{3}$$

The right hand side of this equation is called the *surface term.*

Adjoint Boundary Conditions:

Let the function u satisfy the boundary conditions

$$B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u(b) + \delta_1 u'(b) = 0, \tag{4}$$

$$B_2(u) = \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u(b) + \delta_2 u'(b) = 0, \tag{5}$$

Here $\alpha_i, \beta_i, \gamma_i$, and δ_i are some given constants. If in the above *generalized Green's identity* (3) the function u satisfies the above boundary conditions (4) and (5) then the function v satisfies the *adjoint boundary conditions* when the *surface term vanishes*. Hence for future purposes we can define the *Green's identity*

$$\int_a^b w[\bar{v}\, L_x u - u\, \overline{\{L_x^\dagger\, v\}}]dx = 0, \tag{6}$$

for the functions u satisfying the boundary conditions (4) and (5) and v satisfying the *adjoint boundary conditions*

Self Adjoint Operators:
We shall be interested in second order differential operators

$$L_x u(x) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u, \tag{7}$$

where the coefficients a, b and c are some functions of x. Here we assume that $Re(a) > 0$. It is easy to show that the adjoint operator is given by

$$L^\dagger v = \frac{d^2}{dx^2}(\bar{a}\, v) - \frac{d}{dx}(\bar{b}v) + \bar{c}v, \tag{8}$$

and the function Q in the surface term is found as

$$Q(u, \bar{v}) = a\bar{v}\frac{du}{dx} - u\frac{d}{dx}(a\bar{v}) + bu\bar{v}, \tag{9}$$

If the coefficient functions are real and

$$b(x) = \frac{da(x)}{dx}, \tag{10}$$

Then $L_x^\dagger = L$, becomes a self-adjoint operator and the Q term becomes.

$$Q(u, \bar{v}) = a(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{11}$$

With a suitable weight factor all second order differential operators are self-adjoint

Theorem 1. *Every linear second order differential operator with real coefficients is self adjoint provided the weight function $w(x)$ is chosen properly*

$$w(x) = \frac{p}{a}, \qquad \frac{p'}{p} = \frac{b(x)}{a(x)}, \tag{12}$$

where the differential operator takes the form

$$L_x u = \frac{1}{w}\Big(\frac{d}{dx}\, p(x)\frac{du}{dx}\Big) + cu, \tag{13}$$

and the function Q is given by

$$Q(u, \bar{v}) = p(x)\,\Big(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}\Big), \tag{14}$$

Problems:

1. Prove that the following boundary conditions on $u(x)$, L_x defines a Hermitian differential operator L

(i) $u(a) = u(b) = 0$ (Dirichlet conditons),
(ii) $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (Neumann conditions),
(iii) $\alpha\, u(a) - \frac{du}{dx}(a) = \beta u(b) - \frac{du}{dx}(b)$, ($\alpha, \beta$ real) (general unmixed conditions),
(iv) $u(a) = u(b)$ and $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (periodic conditions).

2. Determine the formal adjoint of each of the operators in (a) through (d) below (i) os a differential operator, and (ii) as an operator, that is, including the boundary conditions (Determine the functions spaces of L and $L^{\dagger}$). Which operators are formally self-adjoint? Which operators are self- adjoint?

(a) $L_x = \frac{d^2}{dx^2} + 1$ in $[0, 1]$ with $u(0) = u(1) = 0$,
(b) $L_x = \frac{d^2}{dx^2}$ in $[0, 1]$ with $u(0) = u'(0) = 0$,
(c) $L_x = \frac{d}{dx}$ in $[0, \infty)$ with $u(0) = 0$,
(d) $L_x = \frac{d^3}{dx^3} - \sin x \frac{d}{dx} + 3$ in $[0, \pi]$ with $u(0) = u'(0) = 0$ and $u''(0) - 4u(\pi) = 0$

Boundary value problems

Let u satisfy the inhomogenous differential equation with homogenous boundary conditions

$$L_x\, u(x) = f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, \tag{15}$$

where $f(x)$ is any integrable function. The function v satisfying the adjoint differential and the homogenous adjoint boundary conditions

$$L_x^\dagger\, v(x) = h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, \tag{16}$$

Green's Functions:
Green's functions $G(x, y)$ and $g(x, y)$ of the above boundary value problems are, respectively given by

$$L_x\, G(x, y) = \frac{\delta(x - y)}{w(x)}, \tag{17}$$

$$L_x^\dagger\, g(x, y) = \frac{\delta(x - y)}{w(x)}, \tag{18}$$

using the Gren's identity for the function u, v, G, and g pairwise (u, v), (u, g), (v, G) and (G, g) we obtain the following important identities

$$u(x) = \int_a^b w(y)\bar{g}(y, x)\, f(y)dy, , \tag{19}$$

$$v(x) = \int_a^b w(y)\bar{G}(y, x)\, h(y)dy, \tag{20}$$

and

$$< f, g >=< G, h >, \quad G(x, y) = \bar{g}(y, x), \tag{21}$$

Since the solutions are given in (19) and (20), then the boundary value problems (15) and (16) reduces to the determination of the Green's functions G and g

Second order operators and Green's functions:
For a second order operators with real coefficients we have

$$\frac{d}{dx}(p(x)\frac{du}{dx}) + cwu = f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, \tag{22}$$
$$\frac{d}{dx}(p(x)\frac{dv}{dx}) + cwv = h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, \tag{23}$$

and the Green's functions satisfy the following equations

$$\frac{d}{dx}(p(x)\frac{dG(x,y)}{dx}) + cwG(x,y) = \delta(x-y), \tag{24}$$
$$\frac{d}{dx}(p(x)\frac{dg(x,y)}{dx}) + cwg(x,y) = \delta(x-y), \tag{25}$$

Since the Greens's functions satisfy the homogenous equation ($L_x G(x,y) = 0$) for $x \neq y$ the we have

$$G(x,y) = \begin{cases} c_1 u_1(x) + c_2 u_2(x) & a \leq x < y \\ d_1 u_1(x) + d_2 u_2(x) & x < y \leq b \end{cases} \tag{26}$$

where u_1 and u_2 are the independent solutions of the homogenous equation $L_x u = 0$ and c_1, c_2, d_1, d_2 are constants (wrt x) to be determined by the following conditions on G. First two are the homogenous boundary conditions and the rest two are the continuity of G and jump condition for the derivative of G

$$B_1(G) = 0, \quad B_2(G) = 0, \tag{27}$$
$$G(x,y)|_{x=y_+} = G(x,y)|_{x=y_+}, \tag{28}$$
$$\frac{dG(x,y)}{dx}|_{x=y_-} - \frac{dG(x,y)}{dx}|_{x=y_-} = \frac{1}{p(y)}, \tag{29}$$

We have the following important theorem.

Theorem 2. *Consider the boundary value problem $L_x u = f(x)$ with the homogenous boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$. Here we assume* *that $L_x$is a second order differential operator. Provided that the homogenous equation has no nontrivial solution satisfying the above boundary conditions the Green's function associated with the boundary value problem exist and unique. The solution is given by*

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y) dy \tag{30}$$

Problems:

3. Let $B_1(u) = 0$ and $B_2(u) = 0$ are defined at $x = a$ and $x = b$ respectively. Solve the Green's function defined in (26).

Solution: The continuity conditions (28) and (29) are solved easily (see DK page 281)

$$G(x,y) = \begin{cases} \frac{U_<(x)U_>(y)}{p(y)W} & a \le x < y \\ \frac{U_<(y)U_>(x)}{p(y)W} & y < x \le b \end{cases} \tag{31}$$

where $W = U_< U_>' - U_> U_<'$ is the Wronksian of $U_<(x) = u_1(x) - \alpha\, u_2(x)$ and $U_>(x) = u_1(x) - \beta u_2(x)$. Here u_1 and u_2 are independent solutions of the homogenous DE (the fundamental set). Prove that $p(x)W(U_<, U_>)$ is constant for all x. The boundary conditions (27) mus also be satisfied, hence

$$B_1(U_<) = B_1(u_1) - \alpha B_1(u_2) = 0,$$
$$B_2(U_>) = B_2(u_1) - \beta B_2(u_2) = 0$$

Hence α and β are determined. As we discussed above the product pW is assumed to be a nonvanishing constant. If it is zero, then $U_<(x) = \sigma U_>(x)$, where σ is an arbitrary constant. Using the definitions of $U_<$ and $U_>$ and α and β found above this relation gives ,

$$B_1(u_1 - \rho u_2) = 0, \quad B_2(u_1 - \rho u_2) = 0$$

We obtain this by letting $\sigma = \rho \frac{B_2(u_2)}{B_2(u_1)}$, where ρ is another constant. The above relation implies the existence of a nontrivial solution $u_1 - \rho u_2$ of the homogenous DE satisfying the homogenous boundary conditions, but this is contradiction with our assumption.

4. Solve the Green's function when both boundary conditions are given at the same point $x = a$. As an example take for instance $u(a) = 0$ and $u'(a) = 0$.

Boundary value problems with inhomogenous boundary conditions: Let $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$, where γ_1 and γ_2 are some given constants. In this case the construction of the Gree's function is exactly similar as above. We determine the Green's function as if the problem is with homogenous boundary conditions. Here the expression for (30) u changes. To obtain the expression for u we use the Green's identity, but in this case since the boundary conditions are homogenous we cannot assume that the surface term is zero. For the operator L_x given in (13) the surface term is given in (14). Now using the generalized Green's identity we get

$$u(x) = \int_a^b w(y)G(x,y)f(y)dy + \{p(y)[u(y)\frac{\partial G(x,y)}{\partial y} - G(x,y)\frac{du}{dy}]\}|_{y=a}^{y=b}, \quad (32)$$

here we use $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$. In the above expression (32) we have $u(a), u(b)$ and $u'(a), u'(b)$ terms in the surface term. Two of them are inserted from the inhomogenous boundary conditions $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$ and the rest dissapears from the expressions due to adjoint homogenous boundary conditions (for self dual operators due to the same homogenous boundary conditions)

Problems:
5. In problem 3 , if $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$ then the complete solution will be

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y)dy + \frac{U_<(x)}{U_<(b}\,\gamma_2 + \frac{U_>(x)}{U_>(a)}\,\gamma_1$$

6. Solve $u'' + u = f(x)$ with $u(0) = u'(0) = 0$.

7. Find the Green's function for $L_x = \frac{d^2}{dx^2} + k^2$ with $u(0) = u(a) = 0$.

8. Find the Green's function for $L_x = \frac{d^2}{dx^2} - k^2$ with $u(\infty) = u(-\infty) = 0$.

9. Find the Geen's function for $L_x = \frac{d}{dx}\, x \frac{d}{dx}$ given the condition that $G(x, y)$ is finite at $x = 0$ and vanishes at $x = 1$.

10. Problem 5 can be generalized further. Let $L_x u = f$ with boundary condition $B_1(u) = \gamma_1$ and $B_2(u) = \gamma_2$. Here L_x is a second order differential operator with real coefficients. Prove that the solution is given by

$$u(x) = \int_a^b w(y)\, G(x, y)\, f(y) dy + \frac{1}{\Delta}[B_2(u_2)\, u_1(x) - B_2(u_1) u_2(x)]\, \gamma_1 + \frac{1}{\Delta}[B_1(u_1) u_2(x) - B_1(u_2) u_1(x)]\, \gamma_2 \quad (33)$$

where $\Delta = B_1(u_1)B_2(u_2) - B_1(u_2)B_2(u_1)$ and the Green's function is found as if the boundary conditions are homogenous. Discuss the case where $\Delta = 0$.

When there exist a nontrivial solution of the homogenous equation satisfying the homogenous boundary conditions:
Let v_i be the nontrivial solutions of the homogenous adjoin DE satisfying adjoint boundary conditions . Hence using the pair (v_i, G) we obtain $v_i = 0$, from (20) by letting $h = 0$ (v satisfies the homogenous equation). To resolve this we must modify the differential equations for G and g. They are

$$\begin{aligned} L_x G(x, y) &= \frac{\delta(x - y)}{w(x)} - \sum_{k=1} \bar{v}_i(x)\, v_i(y), && (34) \\ L_x^\dagger g(x, y) &= \frac{\delta(x - y)}{w(x)} - \sum_{k=1}^{N} \bar{u}_i(x)\, u_i(y) && (35) \end{aligned}$$

where $u_i, (i \leq 2)$, $v_i, (i \leq 2)$ are orthonormalized solutions of homogenous $L_x u = 0$ and $L_x^\dagger v = 0$ satisfying homogenous boundary and adjoint boundary conditions respectively. To restore the uniqueness of the solutions we have the conditions

$$< u_i, G >= 0, (i \leq 2) < v_i, g >= 0 (i \leq 2) \tag{36}$$

For the self adjoint operators and for the case there exists only one homogenous solution satisfying the homogenous boundary conditions we have

$$L_x G(x,y) = -\bar{e}(x)e(y), \quad x \neq y \tag{37}$$

$$\int_a^b G(x,y)\, e(x) dx = 0, \tag{38}$$

here e is normalized solution of the homogenous equation satisfying both homogenous boundary conditions.

See the example in DK page 284. For the solutions to exist we must have (from (21, since $h = 0$)

$$< f, e >= \int_a^b f(x)e(x)dx = 0$$

If this condition is satisfied the solution of boundary value problem is given by

$$u(x) = \alpha e(x) + \int_a^b w(y) G(x,y)\, f(y) dy$$

where $e(x)$ is any solution of the homogenous equation satisfying the homogenous boundary conditions. Hence we have the following theorem

Theorem 3. *Assume that there exists nontrivial solutions of the homogenous DE $Lu = 0$ satisfying the both boundary conditions $B_1(u) =$ and $B_2(u) = 0$ boundary. Then, either the solutions of the boundary value problem $Lu = f$ with $B_1(u) = 0,\ B_2(u) = 0$ do not exist or (if it exists) they are not unique.*

Problems:

11. The solution of the example in DK (page 284) is given as follows: Solution does not exist if $\int_{-a}^{a} f(y)dy \neq 0$. If the equality holds then there are infinitely many solutions of the boundary value problem.

$$u(x) = \alpha + \frac{x}{2a}\int_{-a}^{a} yf(y)dy + \int_{-a}^{x}(x-y)f(y)dy$$

where α is any constant.

12. Solve $u'' + u = f(x)$ with $u(0) = u(\pi) = 0$

13. Solve $u'' = f(x)$ with $u(0) = 0$ and $u'(1) = 1$

Eigenfunction expansion of the Green,s functions

Theorem 4. *Let L_x be second order differential operator and let $L_x u = f(x), x \in [a, b]$ with the boundary conditions $\cos\alpha u(a) - \sin\alpha u'(a) = 0$ and $\cos\beta u(b) + \sin\beta u'(b) = 0$. The we have the following: (i) the spectrum consists entirely of eigenvalues (ii) The eigenvalues are countable and can be listed in a sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $lim_{n\to\infty}\lambda_n = 0$ (iii) the set of normalized eigenfunctions $\{u_i\}$ is an orthonormal basis for $L^2_w(a, b)$ (iv) For the equation $L_x u - \lambda u = f(x)$ exactly one of the following hold:*
(a) If λ is not an eigenvalue of L then the solution is unique for every $f \in L^2_w(a, b)$. This solution is given by

$$u(x) = \sum_{i=1}^{\infty} \frac{(u_i, f >}{\lambda_i - \lambda} u_i(x)$$

(b) If $\lambda = \lambda_j$ is an eigenvalue then

$$u(x) = Cu_j(x) + \sum_{i\neq j}^{\infty} \frac{< u_i, f >}{\lambda_i - \lambda} u_i(x)$$

provided $< f, u_j >= 0$ otherwise there is no solution. Here C is an arbitrary constant

Problems

14. Consider the Sturm-Liouville eigenvalue problem consisting of the differential equations $(L_x u + \lambda u) = 0$ and the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$

(i) $u'' + \lambda u = 0, \quad u(0) = 0, \quad \cos\beta\, u(1) + \sin\beta u'(1) = 0$

(ii) $u'' + \lambda u = 0, \quad u'(0) = 0, \quad \cos\beta\, u(1) + \sin\beta\, u'(1) = 0$
(iii) $(xu')' + \lambda x u = 0, \quad u(a) = u(b) = 0$

Higher dimensional Green's functions:
Let ∇^2 denote the Laplace operator in $n-$ dimensions. Then $\nabla^2 G(\mathbf{x}) = \delta(\mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^n$ has solution (see Set 4, problem 8)

$$G(\mathbf{x}) = \begin{cases} c_n\, r^{2-n} & n \geq 3 \\ c_2 \ln r & n = 2 \end{cases}$$

where

$$\begin{aligned} c_n &= -\frac{1}{(n-2)A_n}, \quad n \geq 3 \\ c_2 &= \frac{1}{2\pi} \end{aligned}$$

Here A_n is the surface area of the unit sphere in n dimensions given by $A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$

More problems about the Green's function technique:

1. Evaluate the Green's function and solutions for each of the following differential equations in the interval $[0, 1]$.

(a) $u'' - k^2 u = f, \quad u(0) - u'(0) = a, u(1) = b,$

(b) $u'' = f, \quad u(0) = u'(0) = 0,$

(c) $u'' + 6u' + 9u = 0, \quad u(0) = 0, u'(0) = 1,$

(d) $u'' + w^2 u = f$, for $x > 0 \quad u(0) = a, u'(0) = 1,$

(e) $u^4 = f, \quad u(0) = 0, y'(0) = 2u'(1), u(1) = a, u''(0) = 0$

2. (From Hildebrand *Advanced Calculus for Applications*)
Consider $u'' + u = f(x)$ with $u(0) = 0$ and $u(a) = 0$ with $\sin a \neq 0$
(a) Show that

$$u(x) = \int_0^x f(y)\sin(x-y)dy + c\,\sin x$$

where c is such that

$$c\,\sin a = -\int_0^a f(y)\sin(a-y)dy$$

Hence

$$u(x) = \frac{1}{\sin a}\left[\int_0^x f(y)\sin(x-y)\,\sin a dy + \int_0^a f(y)\sin(a-y)\sin x\,dy\right]$$

(b) Prove that

$$G(x, y) = \begin{cases} \frac{\sin y \sin(x-a)}{\sin a} & y \leq x \\ \frac{\sin(y-a)\sin x}{\sin a} & x \leq y \end{cases}$$

when $\sin a \neq 0$
(c) If $\sin a = 0$, show that the above equations has no solution unless $f(x)$ satisfies the condition

$$\int_0^a f(x)\sin(a-x)dx = 0$$

in which case there are infinitely many solutions , each of the form

$$u(x) = \int_0^x f(y)\,\sin(x-y)dy + C\sin x$$

where C is an arbitrary constant

3. Solve $u'' = f(x)$ with $u(0) = \alpha$ and $u'(a) = \beta$ and show that

$$u(x) = \alpha + \beta x + \int_0^a G(x,y)f(y)dy$$

with

$$G(x,y) = \begin{cases} -x & x \leq y \\ -y & x \geq y \end{cases}$$

4. Solve $u'' - u = f(x)$ with $u(-\infty) = 0$ and $u(\infty) = 0$ and show that

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\,f(y)dy$$

5. Solve $u'' - \frac{1}{x}u' = f(x)$ with $u(0) = 0$ and $u(1) = 0$ and show that

$$u(x) = \int_0^1 G(x,y)f(y)dy$$

with

$$G(x,y) = \begin{cases} -\frac{(1-y^2)x^2}{2y} & x \leq y \\ -\frac{y(1-x^2)}{2} & x \geq y \end{cases}$$