SET 4

MATH 543: GENERALIZED FUNCTIONS

1. A function $g(x)$ that is differentiable everywhere any number of times is called a *good function* if it and its derivatives vanish as $|x| \to \infty$ faster than any power of $\frac{1}{|x|}$. A function $f(x)$ that is differentiable everywhere any number of times is called a *fairly good function* if its modulus and that of its derivatives does not increase faster than some power of $|x|$ as $|x| \to \infty$. A sequence of good functions $h_n(x)$ defines a *generalized function* $\varphi(x)$ through the relation

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x)\, g(x) dx = \int_{-\infty}^{\infty} \varphi(x)\, g(x) dx$$

where $g(x)$ is any arbitrary good function (test functions). *Regular points* of the generalized functions are those if the sequences defining the generalized functions converge uniformly to an ordinary function in some neighborhood of these points. The limit of the corresponding sequence at these points are called the *local values* of the generalized functions. For instance the sequence $h_n(x) = \sqrt{n/\pi}\, e^{-x^2}$, $(n = 1, 2, \cdots)$ converges uniformly to zero at any point x with $x \neq 0$. Hence the delta function $\delta(x) = 0$ locally for any $x \neq 0$. Prove the following: *If two equivalent sequences (sequences defining the same generalized function) converge uniformly in the neighborhood of $x = x_0$, they determine the same local value of the corresponding generalized function.*

2. Let $a_n(x), b_n(x)$, $(n = 1, 2, \cdots)$ be sequences of good functions defining , respectively the generalized functions $\alpha(x), \beta(x)$. Let $g(x)$ be a good function and $f(x)$ be a fairly good function. Prove the following:
(i) $p\, a_n(x) + q\, b_n(x)$ defines the generalized function $p\alpha(x) + q\beta(x)$ where p and q are arbitrary real numbers.
(ii) $f(x)\, a_n(x)$ defines the generalized function $f(x)\alpha(x)$
(iii) $\frac{da_n(x)}{dx}$ defines the generalized function $\frac{d\alpha(x)}{dx}$.

(iv) Give an example where two sequences $a_n(x)$ and $b_n(x)$ define the generalized functions $\alpha(x)$ and $\beta(x)$ respectively but their product does not converge to a generalized function in the following sense

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} a_n(x) b_n(x) g(x) dx$$

Consider for instance the product sequences in problem 4.

3. Prove the following for the Dirac δ-function:
(i) $\int_a^b \delta(x-c)\, f(x) dx = f(c) \quad$ if $a \leq c \leq b \quad$ otherwise zero.
(ii) $\delta(ax) = \frac{1}{|a|}\, \delta(x)$
(iii) $f(x)\delta(x) = f(0)\, \delta(x)$
(iv) $\delta(f(x)) = \sum_{n=1}^{N} \frac{\delta(x-x_i)}{|df/dx|_{x=x_i}}$, where $f(x)$ has $N-$ roots $x_i, \quad i = 1, 2, \cdots N$
(v) $f(x)\delta'(x) = -f'(0)\, \delta(x)$

4. Prove that each of the following sequences define the Dirac δ-function: (For all cases $n = 1, 2, \cdots$)
(i) $D_n^1 = \frac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$
(ii) $D_n^2 = \frac{1-\cos nx}{\pi n x^2}$
(iii) $D_n^3 = \frac{n}{\pi} \frac{1}{1+n^2x^2}$
(iv) $D_n^4 = \frac{\sin nx}{\pi x}$.

Prove the following:
(a) $\int_{-\infty}^{\infty} D_n^i(x) dx = 1$, $i = 1, 2, 3, 4$ for all n
(b)$\lim_{n\to\infty} \int_{-\infty}^{\infty} D_n^i(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x)\, f(x) dx = f(0)$, $i = 1, 2, 3, 4$.

5. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

(i) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(ii) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$

6. Prove the following:

(i) Let $f(x)$ be a good function then its Fourier transform is also a good function.

(i) Let $f(x)$ be a good function. Then

$$G(t) = \frac{1}{sqrt2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-itx}\, dx$$

implies

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t)\, e^{itx} dx$$

(iii) Let the sequence $f_n(x)$ $(n = 1, 2, \cdots)$ define the generalized function $\varphi(x)$. Then the sequence of of Fourier transforms of the sequence $f_n(x)$

$$F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(x)\, e^{-itx}\, dx$$

defines a generalized function $\Phi(t)$, which is called the Fourier transform of $\varphi(x)$.

7. (a) Let $u_m(x)$, $(m = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. The the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [a, b]$ defines the delta function $\delta(x - x_0)$

(b) Let $u_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{-inx}$, $(n = 0, 1, 2, \cdots)$ with $x \in [-\pi, \pi]$. Prove that the sequence

$$h_n(x) = \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [-\pi, \pi]$ defines also the delta function $\delta(x - x_0)$

You first prove that $\int_I h_n(x) dx = 1$ for all n, where I is the corresponding interval. Then prove that $\lim_{n\to\infty} \int_I f(x)\, h_n(x) dx = f(x_0)$

8. Higher dimensional delta functions can be defined. For instance in three dimensions the delta function $\delta^3(\mathbf{x})$ can be represented in terms one dimensional delta functions. The way we achieve this is to use the general identity $\int_V \delta^3(\mathbf{x}) d^3x = 1$. Hence using this property prove that

(a) In Cartesian coordinates $\delta^3(\mathbf{x}) = \delta(x)\,\delta(y)\,\delta(z)$

(b) In spherical coordinates $\delta^3(\mathbf{x}) = \frac{1}{r^2 \sin\theta}\,\delta(r)\,\delta(\theta)\,\delta(\varphi)$

(c) In cylindrical coordinates $\delta^3(\mathbf{x}) = \frac{1}{\rho}\,\delta(\rho)\,\delta(z)\,\delta(\varphi)$

(d) In three dimensions verify the following identity and find the constant α_3 in the following equation

$$\nabla^2 \frac{1}{|\mathbf{x}|} = \alpha_3\, \delta^3(\mathbf{x})$$

where ∇^2 is the three dimensional Laplace operator (Laplacian).

(e) The above Laplace equation can be written for in any dimension n

$$\nabla^2 \frac{1}{|\mathbf{x}|^{n-2}} = \alpha_n\, \delta^n(\mathbf{x})$$

where $\delta^n(\mathbf{x})$ is the n dimensional δ function and α_n are just constants. Here $n \neq 2$. For $n = 2$ we have

$$\nabla^2 \log|\mathbf{x}| = \alpha_2\, \delta^2(\mathbf{x})$$

Find $\alpha_n, \quad n \neq 2$ and α_2