SET 3

MATH 543: ORTHOGONAL POLYNOMIALS

1. **Generating Functions**: For the classical orthogonal polynomials $C_n(x)$ we have seen so far there exists a generating function $g(x,t)$ for each defined by

$$g(x,t) = \sum_{n=0}^{\infty} a_n \, t^n \, C_n(x), \tag{1}$$

where a_n,s are some real numbers. Find these numbers for the following cases

Plynomial $C_n(x)$	Generating Function $g(x,t)$
Hermite, $H_n(x)$	e^{-t^2+2xt}
Laguerre, $L_n^{\nu}(x)$	$e^{-xt/(1-t)}/(1-t)^{\nu+1}$
Legendre, $P_n(x)$	$(t^2-2xt+1)^{-1/2}$
Chebyshev (I) , $T_n(x)$	$(1-t^2)(t^2-2xt+1)^{-1}$
Chebyshev (II), $U_n(x)$	$(t^2-2xt+1)^{-1}$

2. Find or prove the following for the Legendre polynomials by using its generating function where $a_n = 1$:
(a) $P_n(-x) = (-1)^n P_n(x)$, **(b)**. $||P_n||$, **(c)**. $\int_0^{\infty} P_n(x)dx$,
(d)$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)\, P_n(x) = 0$,
(e) $(1-x^2)\, P_n'(x) + nxP_n(x) - nP_{n-1}(x) = 0$,
(f) $P_{n+1}'(x) - P_{n-1}'(x) - (2n+1)P_n(x) = 0$,
(g) $P_{n+1}' - xP_n' - (n+1)P_n = 0$
(h) Expand $\sin \alpha x$ where $|\alpha| \leq \pi$ and $x \in [-1,1]$ in terms of the Legendre polynomials. Find the first five terms and discuss the convergence of the series expansion

3. Find and prove the following for the Hermite Polynomials by using its generating function where $a_n = \frac{1}{n!}$:
(a) $H_n(-x) = (-1)^n \, H_n(x)$, **(b)** $||H_n||$, **(c)** $H_n(0)$
(d) $H_{n+1} - 2xH_n = 2nH_{n-1}$, **(e)** $H_n'' - 2xH_n' + 2nH_n = 0$,
(f) $\frac{d^m}{dx^m} H_n = 2^m \, \frac{n!}{(n-m)!} H_{n-m}$ (Use the Rodriguez formula to prove this property of the Hermite polynomial).
(g) Expand $e^{-\alpha x^2}$, $\alpha > 0$ in terms of the Hermite polynomials. Find the first five terms. Discuss the convergence of series expansion.

4. Prove the following for the Laguerre Polynomials:
(a) $nL_n^\nu - (n+\nu)L_{n-1}^\nu - xL_n^{\nu\prime} = 0$,
(b) $(n+1)\, L_{n+1}^\nu - (2n + \nu + 1 - x) + (n+\nu)L_{n-1}^\nu = 0$,
(c) Use the generating function to show that $L_n^\nu(0) = \Gamma(n+\nu+1)/[n!\Gamma(\nu+1)]$.
(d) Let $L_n(x) = L_m^0(x)$. Use the generating function for L_n and prove that $L_n'(0) = -n$, $L_n''(0) = \frac{1}{2}n(n-1)$
(e) Expand e^{-kx}, $k > 0$ as a series of Laguerre polynomials $L_n^\nu(x)$ Find the coefficients by using the orthogonality of L_n^ν and the generating function. Discuss the convergence of the series.

5. Prove that Orthogonal Polynomials $C_n(x)$ with $x \in [a,b]$ has n zeros in $[a,b]$.

6. Find $< xC_n, C_m >$ where $C_n(x)$ is any one of the classical orthogonal polynomial with $x \in [a,b]$

7. Express the function $f(x) = -1$ for $-1 \leq x < 0$ and $f(x) = 1$ for $0 < x \leq 1$ in terms of the Legendre polynomials. Find the first five terms and discuss the convergence of the series.

8. For the following problems remember the following theorem:

Theorem: *The Fourier series of a function $f(x)$ that is piecewise continuous in the interval $(-\pi, \pi)$ converges to*

$$\frac{1}{2}[f(x+0)+f(x-0)] \quad for \quad -\pi < x < \pi, \tag{2}$$

$$\frac{1}{2}[f(\pi)+f(-\pi)] \quad for \quad x = \pm\pi \tag{3}$$

(a) Show that

$$|\sin \alpha x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kwx}{4k^2-1}$$

Discuss the above theorem for this example

(b) Find the Fourier series expansion of the function $f(x) = |x|$ in the interval $[-a.a]$. Discuss the above theorem for this example.

(c) Find the Fourier series expansion of the function $f(x) = x + a$ for $-a \leq x < 0$ and $f(x) = x$ for $0 < x \leq a$ and discuss the convergence of the series at the discontinuous points.

9. Prove that the function $f(x) = x$ for $0 \leq x \leq a$ and $f(x) = 2a - x$ for $a \leq x \leq 2a$ has Fourier representation

$$f(x) = \frac{8a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin\frac{n\pi x}{2a} = \frac{8a}{\pi^2}(\sin\frac{\pi x}{2a} - \frac{1}{3^2}\sin\frac{3\pi x}{2a} + \cdots)$$

Discuss the differentiability of the Fourier series. Compare the derivative function f' with the derivative of the Fourier series.

10. If the Fourier expansion

$$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{a} + B_n\sin\frac{n\pi x}{a}) \quad (-a < x < a)$$

show that

$$\frac{1}{a}\int_{-a}^{a}[f(x)]^2dx = 2A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + b_n^2)$$

11. Prove the following

$$\cos \alpha x = \frac{\sin \pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty} (-1)^n \frac{2\alpha \sin \pi\alpha}{\pi(\alpha^2 - n^2)} \cos nx$$

for $-\pi \leq x \leq \pi$. Here α is a non-integer real number. Deduce from this the following formula

$$\cot \pi\alpha = \frac{1}{\pi}[\frac{1}{\alpha} - \sum_{n=1}^{\infty} \frac{2\alpha}{n^2 - \alpha^2}]$$

12. Prove that for $-\pi < x < \pi$

$$e^x = \frac{\sinh \pi}{\pi}[1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + 1}(\cos nx - n \sin nx)]$$

From this result find the Fourier expansion of $\sinh x$ and $\cosh x$ in the same interval

13. (a). Using the Fourier Transform try to give a way to solve the following type of ODE: $y''(x) + ay'(x) + cy(x) = f(x)$ where $y'(0) = \alpha$ and $y(0) = \beta$. Here a, b, c are constants and $f(x)$ is a given function of x.
(b) Find the Fourier Transforms of $1/(1 + x^2)$, $1/(1 + x^2)^2$.