SET 2

MATH 543: ORTHOGONAL FAMILIES AND BASIS

1. Prove that the Fourier coefficients of any $|f> \in L^2_w(a,b)$ form a Hilbert space. First prove that the space of such coefficients form an inner product space then prove that this inner product space is complete. This space is $\mathbf{l_2}$ (see set 1). Prove that l_2 and $L^2_w(a,b)$ are isomorphic. See page 196 of DK.
2. Prove that all finite dimensional inner product spaces are complete. See page 182-183 of DK.
3. Explain the importance of Bessel's inequality and Parseval's relation.
4. Assume that there exists an orthonormal basis $|e_i>$, $(i = 1, 2, \cdots)$ in $L^2_w(a,b)$. Then , for any $|f> \in L^2_{a,b}$, the sequence of vectors

$$|f_k> = \sum_{i=1}^{k} f^i \, |e_i>$$

with

$$f^i = <e_i|f>$$

has $|f>$ as the limit vector in the sense that

$$\lim_{k \to \infty} \rho(|f>, |f_k>) = 0$$

5. Prove that the set of orthonormal vectors $|e_i>$ form a basis of $L^2_e(a,b)$ if and only if the Fourier coefficients wrt $|e_i>$ satisfy the Parseval's relation.
6. Prove that any orthogonal family $|e_i>$, $\; i = 1, 2, = \cdots$ in $L^2_w(a,b)$ is linearly independent
7. If $(e_1, e_2, \cdots, e_n)$ is a finite orthonormal family of functions in $L^2_w(a,b)$ and $f \in L^2_w(a,b)$, then

$$||f - \sum_{k=1}^{n} f^k \, |e_k> ||$$

has its minimum value when $f^k =< f|e_k >$ and

$$\sum_{k=1}^{n} | < f|e_k > |^2 \leq ||f||^2$$

8. Let $f, g \in L^2_w(a,b)$ prove that $| < f|g > | < \infty$.
9. (Gram Schmidt Orthogonalization Process (GS)). Let $|g_i >$, $i = 1, 2, \cdots$ be linearly independent set of vectors in $L^2_w(a,b)$. We can construct an orthogonal set $|u_i >$, $(i = 1, 2 \cdots)$ from the following (prove this statement)

$$|u_1 >= |g_1 >, \quad |u_i >= |g_i > - \sum_{k=1}^{i-1} \alpha_k \, |u_k > \tag{1}$$

where $\alpha_k = \frac{<g_k, u_k>}{||u_k||^2}$ for $k > 1$. Then the orthonormal set is given by $|e_k >= \frac{|u_k>}{||u_k||}$
10. Let $I = [-1, 1]$, $w = 1$ and $(g_i) = (1, x, x^2, \cdots)$ be the linearly independent set in $L^2(-1, 1)$. Find the orthonormal set $|e_i >$ obtained from this linearly independent set. This set is Called the Legendre polynomials.
11. Prove the following proposition: The sequence of orthonormal vectors $\{|e_i >\}$ obtained by the orthogonalization (GS Method) of a linearly independent vectors $|g_i >$ of the space $L^2_w(a,b)$ is a basis of the space if and only if each vector $|f >\in L^2_w(a,b)$ is a limit vector of a sequence of linear combinations of the vectors $|g_i >$.
12. In $L^2_w(a,b)$ any linearly independent set of vectors ($g_i >$'s in the previous problem) may be considered as a basis of this space if an arbitrary vector of $L^2_w(a,b)$ can be expressed as a limiting vector of linear combination of basis vectors. Prove that the set $\{1, x, x^2, \cdots\}$ form a basis of continuous functions in $[a,b]$.
13. Let $f >\in L^2_w(a,b)$ can be expressed as a limiting vector of a sequence of vectors that can be expressed as a linear combination of continuous functions. Prove that , using this assertion and the previous problem, any $|f >\in L^2_w(a,b)$ can be approximated by suitable polynomials. This means

that given $\varepsilon > 0$ there exists a positive integer m and a polynomial p_m such that $||f - p_m|| < \varepsilon$

14. Definition: A complete orthonormal sequence $\{|e_i >, \ i = 1, 2, \cdots\}$ in the space $L^2_w(a, b)$ is called a basis of $L^2_w(a, b)$. Prove the following proposition. Let $\{|e_i >\}, \ i = 1, 2, \cdots\}$ be an orthonormal sequence in $L^2_w(a, b)$. The following statements are equivalent:

1. $\{|e_i >, i = 1, 2, \cdots\}$ is complete.
2. $|f >= \sum_{k=1}^{\infty} < e_k|f > \ |e_i >,$ for all $|f >\in L^2_w(a, b)$
3. $\sum_{k=1}^{\infty} |e_i >< e_k| = \mathbf{1}$.
4. $< f|g >= \sum_{k=1}^{\infty} < f|e_k >< e_k|g >$. for all $|f >, |g >\in L^2_w(a, b)$.
5. $||f||^2 = \sum_{k=1}^{\infty} | < e_k|f > |^2$, for all $|f >\in L^2_w(a, b)$.

Hint:. Use the following directions $1 \to 2 \to 3 \to 4 \to 5 \to 1$.