MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 2, 2008

Thursday 17.40-19.40, SAZ-18

PROBLEMS: Solve three of the following problems. If you solve all problems you will get bonus points.

1. Solve $u'' - \frac{1}{x}u' = f(x)$ with $u(0) = 0$ and $u(1) = 0$ by using the method of Green's function.
Solution: See set.5 (More problems about the Green's function technique, problem 5)

2. Let

$$L(u) = (1 + x^3)u'' + x^4\, u' + (2 + x^5)\, u = 0,$$

and $u(0) = 0$, $u(1) = 0$. Find the adjoint of the operator L and the adjoint boundary conditions.
Solution:

$$\begin{aligned}
L &= (1 + x^3)\, \frac{d^2}{dx^2} + x^4\, \frac{d}{dx} + 2 + x^5 \\
L^\dagger &= \frac{d^2}{dx^2}\, (1 + x^3) - \frac{d}{dx}\, x^4 + 2 + x^5 \\
&= (1 + x^3)\frac{d^2}{dx^2} + (6x^2 - x^4)\, \frac{d}{dx} + 2 + 6x - 4x^3 + x^5, \\
Q(u, v) &= (1 + x^3)(-uv' + vu') + x^4 uv - 3x^2\, uv \qquad (1)
\end{aligned}$$

From the surface term $Q|_{x=0} = Q|_{x=1}$ we get

$$2v(1)u'(1) - v(0)u'(0) = 0$$

Hence the adjoint boundary conditions are $v(1) = 0$ and $v(0) = 0$.

3. Find the general solution of the equation

$$2z^2\, y'' - zy' + (1 + z)y = 0$$

about $z = 0$.
Solution: $r_1 = 1, r_2 = 1/2$

$$y_1(z) = z[1 + \sum_{n\geq 1} \frac{(-1)^n 2^n}{(2n+1)!} z^n]$$

$$y_2(z) = z^{1/2}\,[1 + \sum_{n\geq 1} \frac{(-1)^n 2^n}{(2n)!} z^n]$$

The sums above are convergent for all z (by using the ratio test). The general solution is $y(z) = \alpha y_1(z) + \beta y_2(z)$ where α and β are arbitrary constants.

4. Find the general solution of the equation

$$z^2\, y'' + 3zy' + (1+z)y = 0$$

about $z = 0$.
Solution: $r_1 = r_2 = -1$

$$y_1(z) = \frac{1}{z} \sum_{n\geq 0} \frac{(-1)^n z^n}{(n!)^2}$$

$$y_2(z) = y_1(z)\, \ln z - \frac{2}{z} \sum_{n\geq 1} \frac{(-1)^n \varphi(n)}{(n!)^2} z^n$$

where $\varphi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$ for $n > 0$ and $\varphi(0) = 0$. The sums above are convergent for all z (by using the ratio test). The general solution is $y(z) = \alpha y_1(z) + \beta y_2(z)$ where α and β are arbitrary constants.