MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 28, 2008
Thursday 17.40-19.40, SAZ-18

1. Let S be an infinite dimensional inner product space. State and prove the theorem on Bessel's inequality.

2. Let $P_n(x)$, $(n = 0, 1, 2, \cdots)$, $x \in [-1, 1]$ with the weight function $w = 1$ be the Legendre polynomials where the Rodriquez formula is given by

$$P_n(x) = \frac{(-1)^n}{2^n\, n!} \frac{d^n}{dx^n}(1 - x^2)^n$$

where

$$P_n(x) = k_n x^n + Pol_{n-2}(x), \quad k_n = \frac{(2n)!}{2^n\, (n!)^2}$$

Here $Pol_{n-2}(x)$ is a polynomial of degree less or equal to $n-2$. Furthermore $||P_n||^2 = \frac{2}{2n+1}$. Prove the following:

a)
$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

b)
$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

c)
$$\int_{-1}^{1} P_5(x)\, P_3(x)\, P_2(x) dx = \frac{5}{231}$$

3. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

(a) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(b) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$