MATH 543
METHODS OF APPLIED MATHEMATICS I
Final Exam

January 14, 2008
Monday 12.15-14.15, SAZ02

QUESTIONS: Choose any four out of the following five problems

[25]**1.** Let $z = z_0$ be a regular singular point of a second order differential equation $L(u) = u'' + p(z)z' + q(z)u = 0$ where $p(z)$ and $q(z)$ are continuous functions except the point $z = z_0$. Prove the existence of the solution corresponding the index r_1

[25]**2.** Let $z = z_0$ be a regular singular point of a second order differential equation $L(u) = u'' + p(z)z' + q(z)u = 0$ where $p(z)$ and $q(z)$ are continuous functions except the point $z = z_0$. Suppose that the roots of the r_1 and r_2 of the indical equation differ by an integer $r_1 - r_2 = N$. Prove that the second solution is of the form

$$u(z) = Cu_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} C_n\,(z - z_0)^n$$

where C and C_n are constants.

[25]**3**. Find all regular singular points of the differential equation (Hypergeometric differential equation)

$$z(1 - z)\,u'' + [c - (a + b + 1)\,z]\,u' - ab\,u = 0$$

where a, b and c are real constants. Find the solution of this equation around any one of the these singular points in terms of the hypergeometric function

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0} \frac{\Gamma(a + n)\,\Gamma(b + n)}{\Gamma(c + n)\,\Gamma(n + 1)}\,z^n$$

If you find solutions about all regular singular points you will get 25 extra points from this problem

[25]**4**. Let $y'' + y = \varepsilon y(1 - y'^{\,2})$, for $t > 0$ and $y(0) = 1,\ y'(0) = 0$. Solve this equation up to (including) the first order in ε. Discuss the validity of your approximation.

[25]**5**. Let L be second order hermitian operator and the corresponding boundary value problem be $L(u) = f$ for $x \in (a, b)$ with the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$. Here f is a continuous function in (a, b). Suppose that none of the solutions of the homogeneous problem $L(u) = 0$ satisfies both of the boundary conditions except the trivial solution. Prove that Green's function corresponding to the above boundary value problem exists and unique. Give also the solution of the above boundary value problem.