MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 26, 2007
Thursday 17.40-19.40, SAZ-18

QUESTIONS: Choose any three out of the following four problems

1a. Prove that any orthonormal family $|e_i>$, $i = 1, 2 \cdots$ in $L^2_w(a, b)$ is linearly independent,

1b. Let $|f>$ and $|g>$ be in $L^2_w(a, b)$ prove that $|<f|g>| < \infty$,

1c. Prove that the set of independent vectors $|e_i>$ $(i = 1, 2, ...)$ form a basis of $L^2_w(a, b)$ if and only if the Fourier coefficients with respect to the basis $|e_i>$ satisfy Parseval's relation

2. Define $T_n(x) = \cos(n \cos^{-1} x)$, $n \geq 1$, $T_0 = 1$. Show that

a) $T_n(x)$ is an n-th degree polynomial

b) $\int_{-1}^{1} T_n(x) T_m(x) (1 - x^2)^{-1/2} dx = 0$ when $m \neq n$,
$= \pi/2$ when $m = n$

c) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

d) $(1 - x)\frac{d^2}{dx^2} T_n(x) - x \frac{d}{dx} T_n(x) + n^2 T_n(x) = 0$

3. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

(a) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and

(b) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$

4. Prove the following:
(a) Let $f(x)$ be a good function then its Fourier transform is also a good function.
(b) Let $f(x)$ be a good function. Then

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-itx}\, dx$$

implies

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t)\, e^{itx} dx$$