MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

November 24, 2005
Thursday 08.00-10.30, SAZ21

QUESTIONS:

1a. Let $z = z_0$ be a regular singular point of the differential equation $u'' + p(z)\,u' + q(z)\,u = 0$. Let r_1 and r_2 ($Re(r_1) > Re(r_2)$) be the solutions of the indical equation. When the difference of these indices is a positive integer, prove that the solution $u_2(z)$ corresponding the second index $r = r_2$, near $z = z_0$, is of the form

$$u_2 = C u_1(z)\, \log|z - z_0| + (z - z_0)^{r_2} \sum_0^\infty B_n\,(z - z_0)^n,$$

where u_1 is the solution corresponding to the index $r = r_1$ and C is a constant.

1b. Use the method of Frobenius to find the general solution of the equation

$$z^2\,u'' - 2\,u = 0$$

in the vicinity of the point $z = 0$.

2a. Prove that for a real Hermitian second order operator, the corresponding Green's function $G(x, y)$ is symmetric, $G(x, y) = G(y, x)$.

2b. Let $u_1(x)$ and $u_2(x)$ be two independent solutions of the homogeneous equation $Lu = 0$, where L is second order linear Hermitian operator. Let $G(x, y)$ be the Green's function of the problem

$$L(u) = f(x), \quad a < x < b,$$
$$B_1(u) = 0, \quad B_2(u) = 0$$

with the solution

$$u(x) = \int_a^b G(x, y)\, f(y)\, dy$$

where $B_1(u) = 0$ and $B_2(u) = 0$ are the boundary conditions. *Prove that*

$$u(x) = A\,u_1(x) + B\,u_2(x) + \int_a^b G(x, y) f(y) dy$$

with

$$A = \frac{B_2(u_2)\alpha - B_1(u_2)\beta}{\Omega(u_1, u_2)}, \quad B = \frac{-B_2(u_1)\alpha + B_1(u_1)\beta}{\Omega(u_1, u_2)}$$

and $\Omega(u_1, u_2) = B_1(u_1)\, B_2(u_2) - B_1(u_2)\, B_2(u_1)$, *gives the solution of the inhomogeneous boundary value problem with inhomogeneous boundary values*

$$L(u) = f(x), \quad a < x < b,$$
$$B_1(u) = \alpha, \quad B_2(u) = \beta$$

3a. Use the method of Greens function to solve the following boundary value problem

$$u'' = f(x), \quad a < x < b,$$
$$u'(a) = u'(b) = 0$$

3b. Find the solution of the following initial value problem

$$u'' + z\, u' + u = 0, \tag{1}$$
$$u(0) = 1, \quad u'(0) = 0 \tag{2}$$