MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 20, 2005

Thursday 08.00-10.30, SAZ21

QUESTIONS: Choose any two out of the last three problems

1. Let $P_n(x),\ n = 0, 1, 2, \cdots$ be the Legendre polynomials defined through the Rodriguez formula

$$P_n(x) = \frac{(-1)^n}{2^n\, n!}\, \frac{d^n}{dx^n}\, (1-x^2)^n$$

(Further information: $w = 1, s = 1 - x^2$, and

$$k_n = \frac{2^n\ \Gamma(n+1/2)}{n!\ \Gamma(1/2)},\ k'_n = 0,\ h_n = n + 1/2.$$

(a) Show that

$$\int_{-1}^{1} P_n(x)\, h_m(x)\, dx = 0,\ \ m < n$$

where $h_m(x)$ is a polynomial of degree m. Discuss the case when $m \geq n$.
(b) Prove that $P_n(x)$ is a polynomial of degree n.
(c) Prove the following recursion relation

$$(n+1)P_{n+1} - (2n+1)x\, P_n + nP_{n-1} = 0$$

(d) Prove that

$$(1-x^2)\, \frac{d^2}{dx^2} P_n - 2x\, \frac{d}{dx}\, P_n + n(n+1)P_n = 0$$

2a. Prove that for any good function $f(x)$ and $\epsilon_2 > \epsilon_1$

$$\int_{\epsilon_1}^{\epsilon_2} \delta(x-c)\, f(x)dx = f(c),\ \ \epsilon_1 < c < \epsilon_2$$

2b. Let $h_n(x),\ (n = 0, 1, 2, \cdots)$ be a sequence defined by

$$h_n(x) = \sum_{k=0}^{n} P_k(x)\, P_k(y)$$

where $x, y \in [-1, 1]$ and $P_k(x)$, $(k = 0, 1, 2, \cdots)$ are the Legendre polynomials. Prove that this sequence defines the Dirac $\delta-$ function.

3a. Prove that the Fourier transform of a good function is also a good function.
3b. Prove that the derivative of a generalized function is also a generalized function.