MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

December 17, 2004
Thursday 18.00-20.00, SAZ-18

QUESTIONS: Choose any three out of the following four problems
1. Let $\frac{d^2u}{dz^2} + p(z)\,\frac{du}{dz} + q(z)\,u = 0$ in a region R of the complex plane where the functions p and q are analytic except some enumerable number of points of R where these functions may have isolated singularities. Let z_0 be an ordinary point of the this differential equation so that $u(z_0) = u_0$ and $\frac{du}{dz}(z_0) = u_1$ where u_0 and u_1 are some given constants. Prove that this initial value problem has a unique solution in the neighborhood of z_0.

(**2a**). Solve the series solution the differential equation $2u'' + zu' + 3u = 0$ about the point $z = 0$.
Solution: $z = 0$ is an ordinary point of the DE. Hence power series expansion $u(z) = \sum_0^\infty a_n\, z^n$ should give the solution about $z = 0$. It is given by $u(z) = \alpha\, u_1(z) + \beta\, u_2(z)$ where α and β are arbitrary constants and

$$\begin{aligned} u_1(z) &= 1 - \frac{3}{4}z^2 + \frac{5}{32}\, z^4 + \cdots + (-)^n \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{2^n\,(2n)!} z^{2n} \cdots, \\ u_2(z) &= x - \frac{1}{3}\, z^3 + \frac{1}{20}\, x^5 + \cdots + (-1)^n\, \frac{4 \cdot 6 \cdots (2n+2)}{2^n\,(2n+1)!}\, z^{2n+1} + \cdots \end{aligned}$$

The coefficients a_n satisfy the recursion relation $2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0$.
(**2b**). Solve the differential equation $2z^2\, u'' - zu' + (1+z)\, u = 0$ about its regular singular points.
Solution: $z = 0$ is the regular singular point of the DE with $a_0 = -1/2,\ b_0 = 1/2$. Hence $r_1 = 1,\ \ r_2 = 1/2$ and

$$a_{n+1} = \frac{a_{n-1}}{(r+n-1)(2r+2n-1)} \quad n \geq 1$$

which leads to

$$\begin{aligned} r_1 &= 1, \quad a_n = \frac{(-1)^n}{[1 \cdot 3 \cdot 5 \cdots (2n+1)]n!}\, a_0, \quad n \geq 1, \\ r_2 &= 1/2, \quad a_n = \frac{(-1)^n}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]n!}\, a_0, \quad n \geq 1 \end{aligned}$$

and the corresponding solutions are

$$u_1(z) = z\,[1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}], \qquad (1)$$

$$u_2(z) = \sqrt{z}\,[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, z^n}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]n!}] \qquad (2)$$

The general solution is $u(z) = \alpha u_1(z) + \beta u_2(z)$ where α and β are arbitrary constants.

(**3a**). Prove that a second order FDE having only two singular points is equivalent to a DE with constant coefficients, hence solvable in terms of the elementary functions $\sin z$,$\cos z$ and polynomial in z.

(**3b**). *The Riemann Equation:*. A second order linear FDE with (only) three regular singular points $z = z_1$, $z = z_2$ and $z = z_3$ is given in the form

$$u'' + \left(\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right) u' + \left(\frac{(z_1-z_2)(z_1-z_3)\alpha\,\alpha'}{z-z_1} + \frac{(z_2-z_1)(z_2-z_3)\beta\beta'}{z-z_2} + \frac{(z_3-z_1)(z_3-z_2)\gamma\,\gamma'}{z-z_3}\right) \frac{u}{(z-z_1)(z-z_2)(z-z_3)} = 0,$$

where α, α', β, β', γ, γ' are constants. Prove that z_1 z_2 and z_3 are regular singular points of the Riemann equation and find the indices corresponding to these regular singular points.

(**3c**). z_1 , z_2 and z_3 are the only singular points of the above differential equation. Prove that

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

(**4**). The Hypergeometric equation is given by

$$z(1-z)u'' + [c - (a+b+1)z]\, u' - abu = 0$$

where a, b, c are some constants and $u(z) = F(a, b; c; z)$. Here F denotes the hypergeometric function.

(**4a**). Find the solution of this equation about $z = 0$.

(**4b**). Prove that $\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-z^2\sin^2\theta}} = F(1/2,1/2;1;z^2)$

(**4c**). Find differential equation satisfied by the function

$$\Phi = \lim_{a\to\infty} F(a,b;c;z/a)$$