MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

**November 09, 2004**
Thursday 18.00-20.00, SAZ-18

**QUESTIONS: Choose any three out of the following four problems**

**1a**. Prove that any orthonormal family $|e_i>$, $i = 1, 2\cdots$ in $L^2_w(a,b)$ is linearly independent,
**1b**. Let $|f>$ and $|g>$ be in $L^2_w(a,b)$ prove that $|<f|g>| < \infty$,
**1c**. Prove that the set of independent vectors $|e_i>$ $(i = 1, 2, ...)$ form a basis of $L^2_w(a,b)$ if and only if the Fourier coefficients with respect to the basis $|e_i>$ satisfy Parseval's relation

**2a**. Give a definition of a *bases* in a vector space and explain its importance,
**2b**. Explain the importance of *completeness* in a vector space (in particular in $L^2_w(a,b)$),
**2c**. Let $L$ be differential operator, of order $n$, acting on a function space $U$. Let $Lu = f$ defines a boundary value problem in $[a,b]$ with the boundary conditions $B_i(u) =$, $i = 1, 2, \cdots, n-1$. Define the *adjoint boundary value problem.*

**3**. Hermite polynomials $H_n(x)$, $(n = 0, 1, 2\cdots)$ where $x \in (-\infty, \infty)$ are defined through

$$H_n(x) = (-1)^n \, e^{x^2} \, \frac{d^n}{dx^n} \, e^{-x^2}$$

with $w(x) = e^{-x^2}$, $s(x) = 1$, $k_n = 2^n$, $k' = 0$ and $h_n = 2^n \, n! \, \sqrt{\pi}$.
**3a**. Prove that

$$\frac{d^2}{dx^2} \, H_n(x) - 2x \, \frac{d}{dx} \, H_n(x) + 2n H_n(x) = 0$$

**3b**. Prove that

$$H_{n+1}(x) = 2x \, H_n(x) - 2n H_{n-1}(x)$$

**3c**. Prove that $H_n$'s are orthogonal to any polynomial of order less or equal to $n-1$.

**4a**. The sequence $D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$, $(n = 1, 2, \cdots)$ defines a distribution over $(\infty, \infty)$ called the Dirac $\delta$-function $\delta(x)$. Prove that for all *good functions* $f(x)$ and for all $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} f(x)\, \delta(x)\, dx = f(0)$$

**4b**. Find the fourier transform of $\delta(x)$

**4c**. Find the solution of

$$u''(x) + k^2\, u(x) = \delta(x - a), \quad x \in [0, c]$$

where $c > a > 0$ with $u(0) = 0$ and $u'(c) = 0$ (Here $ka \neq 0, \pm\pi, \pm 2\pi, \cdots$)