MATH 543

METHODS OF APPLIED MATHEMATICS I

Final Exam

January 21, 2004

Wednesday 09.00-11.30, SBZ-03

QUESTIONS: Solve two of the first three and three of the last six problems. Total number of problems you solve must be five

Let $C_n(x)$ be the classical polynomials defined by the Rodriguez formula

$$C_n(x) = \frac{1}{w}\frac{d^n}{dx^n}(ws^n), \quad n = 0, 1, 2, \cdots$$

where $x \in [a, b]$, $C_1(x)$, $w = w(x)$ and $s = s(x)$ satisfy the following conditions

(i) $C_1(x)$ is a first degree polynomial in x

(ii) $s(x)$ is a polynomial in of degree ≤ 2 with real roots,

(iii) $w(x)$ is a real , positive and integrable function in $[a, b]$ and satisfy the boundary conditions

$$w(a)\, s(a) = w(b)\, s(b) = 0$$

1. Show that **(a)** $s(x)$ can not have imaginary roots **(b)** the roots of $s(x)$ cannot be the same

2. Let $P_m(x)$ be a polynomial of degree m prove that

$$\int_a^b w(x)\, C_n(x)\, P_m(x)\, dx = 0, \quad m < n$$

3. **(a)** Let $u_n(x)$, $(n = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. Prove that the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [a, b]$ defines the delta function $\delta(x - x_0)$. **(b)** Let

$$u_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{-inx}, \quad (n = 0, 1, 2, \cdots)$$

with $x \in [-\pi, \pi]$. Prove that the sequence

$$h_n(x) = \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [-\pi, \pi]$ defines also the delta function $\delta(x - x_0)$

4. Let A and B two points on a right cylinder in $\mathbf{R}^3$, i.e., $x^2 + y^2 = r_0^2$ where r_0 is the radius. Find the curve (or curves) C connecting these two points so that its length is minimum. (Hint: use cylindrical coordinates for the length formula)

5. Find the extremals of the functional

$$J(y) = \int_0^1 [(y')^2 + x^2]\, dx$$

so that $y(0) = 0, \quad y(1) = 0$ with the constraint

$$\int_0^1 y^2\, dx = 2$$

6. Let

$$y'' + y = \varepsilon\, y(y')^2, \quad t > 0$$

where $y(0) = 1$ and $y'(0) = 0$ and $0 < \varepsilon << 1$. Find a two term (uniformly valid) perturbation approximation. Discuss the convergence of the perturbation series (whether the approximate solution is a uniformly valid one).

7. Use the singular perturbation method to obtain a uniform approximate solution to the following problem.

$$\varepsilon\, y'' + (t + 1)\, y' + y = 0, \quad 0 < t < 1$$

where $y(0) = 0$, $y(1) = 1$ and $0 < \varepsilon << 1$. Verify that your solution is a uniformly valid solution.

8. Find a two term approximation for large λ for the integral

$$I(\lambda) = \int_0^{\pi/2} e^{-\lambda \tan^2 u}\, du$$

9. Find an integral representation of the solutions of the DE

$$(1-z)^2\, u'' + 3(1-z)\, u' - u = 0$$

Some useful formulas:

$$\Gamma(z) = \int_0^\infty u^{z-1}\, e^{-u}\, du, \quad z > 0, \quad \Gamma(z+1) = z\,\Gamma(z), \quad \Gamma(1/2) = \sqrt{\pi}$$