ERROR ANALYSIS OF THE GENERALIZED MAC SCHEME

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Abstract.
We present a rigorous convergence analysis for the generalized MAC (GMAC) scheme on curvilinear domains proposed earlier by the authors [HLW]. The error estimate for the velocity field is established by energy estimate utilizing the stream function and discrete identities associated with the spatially compatible discretization. The spatially compatible discretization also induces subtle stabilizing effect that renders the scheme uniform LBB bound even though GMAC is staggered and supported the same way as the $Q_1 - P_0$ finite element method, which is well known to be unstable under divergence constraint. As a result, full second order error estimate is achieved for both velocity and pressure with minimal regularity requirement.

1. Introduction. A generalized MAC scheme (GMAC) for the Navier-Stokes equation in rotational form

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p &= \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f} && \text{on } \Omega \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{on } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma \end{aligned} \tag{1.1}$$

on curvilinear domains was introduced in [HLW]. With partially staggered grids (velocity components collocated on cell centers, pressure placed on grid points) and centered difference in a locally 'skewed' coordinate, the discretization preserves crucial identities such as

$$\mathrm{curl}_h \circ \mathrm{grad}_h \equiv \boldsymbol{0}, \qquad \mathrm{div}_h \circ \mathrm{curl}_h \equiv 0 \tag{1.2}$$

and their converse in the discrete setting. The resulting scheme is simple and efficient with full second order accuracy on curvilinear domains. A key ingredient of the scheme is the proper treatment at the boundary, which not only enforces the pressure as discrete Lagrangian multiplier without introduction artificial boundary conditions, but also leads to an *exact* Hodge decomposition for the velocity field, which plays a key role in both stability and efficiency of the scheme.

In this paper, we will show that the spatial compatibility (1.2) (recast as Lemma 2.1 in section 2) also leads to a simple error estimate for the velocity field. Optimal error analysis for the classical MAC scheme was first obtained in [HW], and in [W1] for MAC-like schemes on Cartesian grids. The proof in [HW, W1] is based on high order Strang's expansion. Here we present an alternative approach that relies mainly on the special structure of the spatial discretization, making use of both the stream function and the discrete differential identities in Lemma 2.1. As a result, we obtain optimal $O(h^2)$ error estimate for the velocity field provided the exact solution satisfies $\boldsymbol{u}^e \in L^2(0,T;C^4(\bar{\Omega})) \cap H^1(0,T;C^2(\bar{\Omega}))$ and $p^e \in L^2(0,T;C^3(\bar{\Omega}))$ (Theorem 1, section 3.1). This may be the minimal regularity requirement in finite difference setting.

On the other hand, the error analysis for the pressure is much more subtle due to lack of evolutionary equation. Our approach is based on establishing the Ladyzhenskaya-Babuška-Brezzi condition (also known as the LBB, div-stability or inf-sup condition) for the generalized MAC scheme. The LBB condition provides direct access to pressure error estimate for the dynamic problem (1.1), and is essential to the solvability and uniform estimate for the static Stokes problem. It is worth noting that GMAC is staggered the same way as the $Q_1 - P_0$ finite element method, which is known to violate the uniform LBB condition. The main difference between the two schemes is the discretization of the viscous term. The spatially compatible discretization associated with GMAC induces subtle stabilizing effect and is key to the uniform LBB bound. As a result, we

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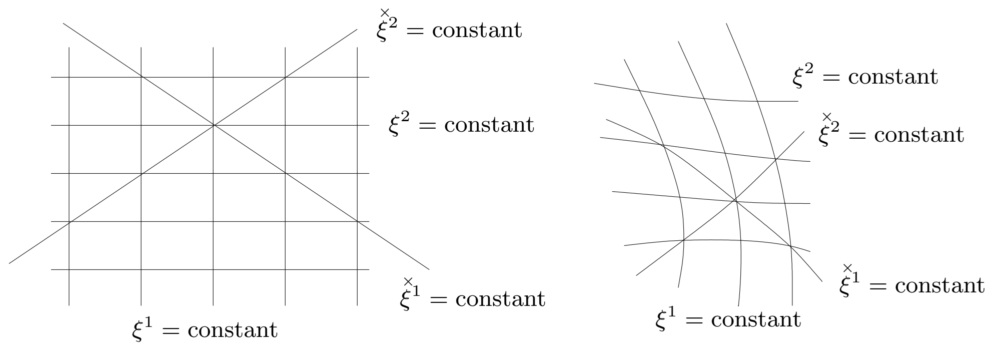

FIG. 2.1. *The computational domain (left) and physical domain (right)*

obtain second order pressure estimate with $\boldsymbol{u}^e \in L^2(0,T;C^4(\bar{\Omega})) \cap H^1(0,T;C^3(\bar{\Omega}))$ and $p^e \in L^2(0,T;C^3(\bar{\Omega}))$ (Theorem 3, section 3.2).

The rest of the paper is organized as follows. In section 2 we review the generalized MAC scheme, the boundary treatment and relevant identities associated with the spatial discretizations. In section 3.1, we proceed with the basic energy estimate and give a second order error estimate for the velocity and a first order error estimate for the pressure gradient. The full second order pressure estimate utilizing the LBB condition is given in section 3.2. Finally, the technical proof of the LBB condition is given in the Appendix.

2. Generalized MAC Scheme in Curvilinear Coordinate. We briefly summarize the spatial discretization of the generalized MAC scheme in the 2D setting. The details can be found in [HLW]. Denote by $\boldsymbol{x} = (x,y)$ the position vector in the physical domain and (ξ^1, ξ^2) the coordinate in the computational domain with mesh size $\Delta\xi^1 = h_1$ and $\Delta\xi^2 = h_2$. Instead of the default coordinate (ξ^1,ξ^2), a local coordinate in the skewed direction (Figure 2.1)

$$\overset{\times}{\xi}{}^1 \triangleq \frac{h_2\xi^1 + h_1\xi^2}{\sqrt{h_1^2+h_2^2}}, \qquad \overset{\times}{\xi}{}^2 \triangleq \frac{-h_2\xi^1 + h_1\xi^2}{\sqrt{h_1^2+h_2^2}}. \tag{2.1}$$

is selected to realize the discretization of (1.1).

With the skewed local coordinate, the differential operators can be determined following standard procedure. The superscript '×' is used to denote quantities computed in the skewed variables $\overset{\times}{\xi}{}^\alpha$. The basis vectors and metric tensors with respect to the skewed coordinate $(\overset{\times}{\xi}{}^1, \overset{\times}{\xi}{}^2)$ are thus denoted by

$$\overset{\times}{\boldsymbol{e}}_1 = \frac{\partial \boldsymbol{x}}{\partial \overset{\times}{\xi}{}^1}, \qquad \overset{\times}{\boldsymbol{e}}_2 = \frac{\partial \boldsymbol{x}}{\partial \overset{\times}{\xi}{}^2}, \qquad \overset{\times}{g}_{\mu\nu} = \overset{\times}{\boldsymbol{e}}_\mu \cdot \overset{\times}{\boldsymbol{e}}_\nu, \tag{2.2}$$

$$\overset{\times}{\boldsymbol{e}}{}^1 = \nabla \overset{\times}{\xi}{}^1, \qquad \overset{\times}{\boldsymbol{e}}{}^2 = \nabla \overset{\times}{\xi}{}^2, \qquad \overset{\times}{g}{}^{\mu\nu} = \overset{\times}{\boldsymbol{e}}{}^\mu \cdot \overset{\times}{\boldsymbol{e}}{}^\nu. \tag{2.3}$$

The Jacobian of the transformation between $\boldsymbol{x}$ and $\overset{\times}{\boldsymbol{\xi}}$ is given by

$$\sqrt{\overset{\times}{g}} = \det\left(\frac{\partial \boldsymbol{x}}{\partial \overset{\times}{\boldsymbol{\xi}}}\right) = \sqrt{\det(\overset{\times}{g}_{\mu\nu})}. \tag{2.4}$$

The covariant and contra-variant components of a vector field $\boldsymbol{u}$ are defined through

$$\boldsymbol{u} = \overset{\times}{u}{}^1 \overset{\times}{\boldsymbol{e}}_1 + \overset{\times}{u}{}^2 \overset{\times}{\boldsymbol{e}}_2 = \overset{\times}{u}_1 \overset{\times}{\boldsymbol{e}}{}^1 + \overset{\times}{u}_2 \overset{\times}{\boldsymbol{e}}{}^2. \tag{2.5}$$

The transformation between the covariant and contra-variant components for the metric tensor and for a vector field are given by

$$\sum_{\gamma=1}^{2} \overset{\times}{g}^{\mu\gamma} \overset{\times}{g}_{\gamma\nu} = \delta^{\mu}_{\nu} \tag{2.6}$$

and

$$\overset{\times}{u}^{\mu} = \sum_{\gamma=1}^{2} \overset{\times}{g}^{\mu\gamma} \overset{\times}{u}_{\gamma}, \qquad \overset{\times}{u}_{\nu} = \sum_{\gamma=1}^{2} \overset{\times}{g}_{\gamma\nu} \overset{\times}{u}^{\gamma}, \qquad \mu,\nu = 1,2. \tag{2.7}$$

The discretization of (1.1) is based on centered difference approximation of the intrinsic differential operators given below in (2.19)-(2.23). The metric tensors involved there can be calculated either analytically, given explicit form of the mapping $(\xi^1, \xi^2) \mapsto (x, y)$, or numerically from centered difference approximation of (2.2) for the covariant component, and then from (2.4) and (2.6) for the numerical Jacobian and contra-variant components.

Here for simplicity of presentation, we assume the physical domain Ω is diffeomorphic to a ring, so that a single coordinate chart $(\xi^1, \xi^2) \in (0,1) \times S^1$ is sufficient to represent the computational domain. The generalized MAC scheme can be applied to a generic domain by decomposing it into non-overlapping quadrilateral sub-domains. The details of the discretizations on coordinate interfaces and junctions, as well as the 3D case can be found in [HLW].

We further assume equal spacing in ξ^1 and ξ^2:

$$h_1 = h_2 = h = \frac{\overset{\times}{h}}{\sqrt{2}} = \frac{1}{N}, \tag{2.8}$$

where $\overset{\times}{h} = \Delta\overset{\times}{\xi}^1 = \Delta\overset{\times}{\xi}^2 = \frac{2h_1h_2}{\sqrt{h_1^2+h_2^2}}$ is the natural grid spacing in the skewed variables. When $h_1 = h_2$, we also have $\sqrt{g} = \sqrt{\overset{\times}{g}}$. The relevant domains for spatial discretizations are summarized as follows:

$$\Omega_{\mathrm{c}} \triangleq \left\{ \boldsymbol{x}(\xi^1_{i-\frac{1}{2}}, \xi^2_{j-\frac{1}{2}}) \,|\, 1 \le i \le N;\, 1 \le j \le N \right\} \tag{2.9}$$

$$\mathring{\Omega}_{\mathrm{g}} \triangleq \left\{ \boldsymbol{x}(\xi^1_{i}, \xi^2_{j}) \,|\, 1 \le i \le N-1;\, 1 \le j \le N \right\} \tag{2.10}$$

$$\bar{\Omega}_{\mathrm{g}} \triangleq \left\{ \boldsymbol{x}(\xi^1_{i}, \xi^2_{j}) \,|\, 0 \le i \le N;\, 1 \le j \le N \right\} \tag{2.11}$$

$$\Gamma_{\mathrm{g}} \triangleq \bar{\Omega}_{\mathrm{g}} \setminus \mathring{\Omega}_{\mathrm{g}} \tag{2.12}$$

$$\bar{\Omega}_{\mathrm{g_e}} \triangleq \left\{ \boldsymbol{x}(\xi^1_{i}, \xi^2_{j}) \in \bar{\Omega}_{\mathrm{g}} \,|\, i+j \text{ is even} \right\} \tag{2.13}$$

$$\bar{\Omega}_{\mathrm{g_o}} \triangleq \left\{ \boldsymbol{x}(\xi^1_{i}, \xi^2_{j}) \in \bar{\Omega}_{\mathrm{g}} \,|\, i+j \text{ is odd} \right\} \tag{2.14}$$

By assumption, all scalar and vector fields under consideration, including

$$L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R}) \triangleq \left\{\omega : \bar{\Omega}_{\mathrm{g}} \to \mathbb{R}\right\} \tag{2.15}$$

$$L^2(\Omega_{\mathrm{c}}, \mathbb{R}^2) \triangleq \left\{\boldsymbol{u} : \Omega_{\mathrm{c}} \to \mathbb{R}^2\right\} \tag{2.16}$$

and

$$L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2 \triangleq \Big\{p \in L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R}) \,\Big|\, \sum_{\bar{\Omega}_{\mathrm{g_e}}}{}'(\sqrt{\overset{\times}{g}_h}p)_{i,j} = 0 = \sum_{\bar{\Omega}_{\mathrm{g_o}}}{}'(\sqrt{\overset{\times}{g}_h}p)_{i,j} \Big\}, \tag{2.17}$$

$$L^2_c(\bar{\Omega}_{\mathrm{g}}, \mathbb{R}) \triangleq \Big\{\psi \in L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R}) \,\Big|\, \psi_{i,j} = \text{constant on } i=0 \text{ and } i=N, \text{ respectively}\Big\} \tag{2.18}$$

are all periodic in ξ^2. Here in (2.17), the primed sums denote summing with half weight on boundary grids $\Gamma_{\rm g} \cap \bar{\Omega}_{\rm g_e}$ and $\Gamma_{\rm g} \cap \bar{\Omega}_{\rm g_o}$ respectively.

For $\omega, p \in L^2(\bar{\Omega}_{\rm g}, \mathbb{R})$ and $\boldsymbol{u} = \overset{\times}{u}{}^1 \overset{\times}{\boldsymbol{e}}_1 + \overset{\times}{u}{}^2 \overset{\times}{\boldsymbol{e}}_2 = \overset{\times}{u}_1 \overset{\times}{\boldsymbol{e}}{}^1 + \overset{\times}{u}_2 \overset{\times}{\boldsymbol{e}}{}^2 \in L^2(\Omega_{\rm c}, \mathbb{R}^2)$, we define

$$\overset{\times}{\nabla}_h : L^2(\bar{\Omega}_{\rm g}, \mathbb{R}) \mapsto L^2(\Omega_{\rm c}, \mathbb{R}^2), \qquad \overset{\times}{\nabla}_h p \triangleq (\overset{\times}{D}_1 p)\, \overset{\times}{\boldsymbol{e}}{}^1 + (\overset{\times}{D}_2 p)\, \overset{\times}{\boldsymbol{e}}{}^2 \tag{2.19}$$

$$\overset{\times}{\nabla}{}^{\perp}_h : L^2(\bar{\Omega}_{\rm g}, \mathbb{R}) \mapsto L^2(\Omega_{\rm c}, \mathbb{R}^2), \qquad \overset{\times}{\nabla}{}^{\perp}_h \omega \triangleq \frac{-\overset{\times}{D}_2 \omega}{\sqrt{\overset{\times}{g}_h}} \overset{\times}{\boldsymbol{e}}_1 + \frac{\overset{\times}{D}_1 \omega}{\sqrt{\overset{\times}{g}_h}} \overset{\times}{\boldsymbol{e}}_2 \tag{2.20}$$

$$\overset{\times}{\nabla}{}'_h \cdot : L^2(\Omega_{\rm c}, \mathbb{R}^2) \mapsto L^2(\bar{\Omega}_{\rm g}, \mathbb{R}), \qquad \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u} = \frac{1}{\sqrt{\overset{\times}{g}_h}} \big(\overset{\times}{D}{}'_1 (\sqrt{\overset{\times}{g}_h}\, \overset{\times}{u}{}^1) + \overset{\times}{D}{}'_2 (\sqrt{\overset{\times}{g}_h}\, \overset{\times}{u}{}^2) \big) \tag{2.21}$$

$$\overset{\times}{\nabla}{}^{\perp\prime}_h \cdot : L^2(\Omega_{\rm c}, \mathbb{R}^2) \mapsto L^2(\bar{\Omega}_{\rm g}, \mathbb{R}), \qquad \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \boldsymbol{u} = \frac{1}{\sqrt{\overset{\times}{g}_h}} (\overset{\times}{D}{}'_1 \overset{\times}{u}_2 - \overset{\times}{D}{}'_2 \overset{\times}{u}_1) \tag{2.22}$$

and

$$\overset{\times}{\triangle}{}'_h : L^2(\bar{\Omega}_{\rm g}, \mathbb{R}) \mapsto L^2(\bar{\Omega}_{\rm g}, \mathbb{R}), \qquad \overset{\times}{\triangle}{}'_h p = \frac{1}{\sqrt{\overset{\times}{g}_h}} \sum_{\mu,\nu=1}^{2} \overset{\times}{D}{}'_\mu (\sqrt{\overset{\times}{g}_h} \overset{\times}{g}{}^{\mu\nu}_h \overset{\times}{D}_\nu p). \tag{2.23}$$

The primed operators in (2.21)-(2.23) denote the 'reduced' operators following Anderson [AN]. The reduction only takes place near boundary, where all quantities involving the metric tensors located outside the computational domain are set to zero, followed by proper normalization. For example, at interior grids $1 < i < N$, (2.23) gives the full Laplacian

$$\begin{aligned}(\overset{\times}{\triangle}{}'_h p)_{i,j} = (\overset{\times}{\triangle}_h p)_{i,j} = \frac{1}{\sqrt{\overset{\times}{g}_{h_{i,j}}}} \Bigg(& \frac{(\overset{\times}{q}{}^{11}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{12}_h \overset{\times}{D}_2 p)_{i+\frac12, j+\frac12}}{\overset{\times}{h}} + \frac{(\overset{\times}{q}{}^{21}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{22}_h \overset{\times}{D}_2 p)_{i-\frac12, j+\frac12}}{\overset{\times}{h}} \\ & - \frac{(\overset{\times}{q}{}^{11}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{12}_h \overset{\times}{D}_2 p)_{i-\frac12, j-\frac12}}{\overset{\times}{h}} - \frac{(\overset{\times}{q}{}^{21}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{22}_h \overset{\times}{D}_2 p)_{i+\frac12, j-\frac12}}{\overset{\times}{h}} \Bigg)\end{aligned} \tag{2.24}$$

where $\overset{\times}{q}{}^{\alpha\beta}_h = \sqrt{\overset{\times}{g}_h} \overset{\times}{g}{}^{\alpha\beta}_h$. At a boundary grid, say $i = 0$, the discrete Laplacian reduces to

$$(\overset{\times}{\triangle}{}'_h p)_{0,j} = \frac{2}{\sqrt{\overset{\times}{g}_{h_{0,j}}}} \left(\frac{(\overset{\times}{q}{}^{11}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{12}_h \overset{\times}{D}_2 p)_{\frac12, j+\frac12}}{\overset{\times}{h}} - \frac{(\overset{\times}{q}{}^{21}_h \overset{\times}{D}_1 p + \overset{\times}{q}{}^{22}_h \overset{\times}{D}_2 p)_{\frac12, j-\frac12}}{\overset{\times}{h}} \right). \tag{2.25}$$

The detailed formula of (2.19)-(2.23) can be found in [HLW]. It can be shown that

$$\ker(\overset{\times}{\triangle}{}'_h) = \ker(\overset{\times}{\nabla}_h) = \operatorname{span}\{\mathbf{1}_{\bar{\Omega}_{\rm g_e}}, \mathbf{1}_{\bar{\Omega}_{\rm g_o}}\} \tag{2.26}$$

In case N is odd, $\bar{\Omega}_{\rm g_e}$ and $\bar{\Omega}_{\rm g_o}$ coincide due to periodicity in ξ^2. To be definite, we assume without loss of generality that N is even and therefore $\dim\big(\ker(\overset{\times}{\triangle}{}'_h)\big) = \dim\big(\ker(\overset{\times}{\nabla}_h)\big) = 2$.

The significance of the reduced operator can be seen from the role it plays in the adjointness with respect to the natural inner products:

$$
\begin{aligned}
\langle \boldsymbol{u} \,,\, \boldsymbol{v} \rangle_{\Omega_\mathrm{c}} &= h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \big((\boldsymbol{u}\cdot\boldsymbol{v})\sqrt{g}_h \big)_{i-\frac12,j-\frac12} = h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \big((\overset{\times}{u}_1 \overset{\times}{v}{}^1 + \overset{\times}{u}_2 \overset{\times}{v}{}^2)\sqrt{g}_h \big)_{i-\frac12,j-\frac12} \\
&= h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \big((\overset{\times}{u}{}^1 \overset{\times}{v}_1 + \overset{\times}{u}{}^2 \overset{\times}{v}_2)\sqrt{g}_h \big)_{i-\frac12,j-\frac12}, \qquad \boldsymbol{u},\, \boldsymbol{v} \in L^2(\Omega_\mathrm{c}, \mathbb{R}^2),
\end{aligned} \tag{2.27}
$$

$$
\langle a \,,\, b \rangle_{\bar{\Omega}_\mathrm{g}} = h^2 {\sum_{i=0}^{M}}' \sum_{j=1}^{N} \big(a\, b \sqrt{g}_h \big)_{i,j}, \qquad a,\, b \in L^2(\bar{\Omega}_\mathrm{g}, \mathbb{R}). \tag{2.28}
$$

More precisely, we have the following Lemma from [HLW] which plays an essential role in the error analysis to be presented below:

Lemma 2.1. *Let $\boldsymbol{u} \in L^2(\Omega_\mathrm{c}, \mathbb{R}^2)$ and $a \in L^2(\bar{\Omega}_\mathrm{g}, \mathbb{R})$, we have*

1.
$$
\langle \boldsymbol{u} \,,\, \overset{\times}{\nabla}_h a \rangle_{\Omega_\mathrm{c}} = -\langle \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u} \,,\, a \rangle_{\bar{\Omega}_\mathrm{g}} \tag{2.29}
$$

2.
$$
\langle \boldsymbol{u} \,,\, \overset{\times}{\nabla}{}^{\perp}_h a \rangle_{\Omega_\mathrm{c}} = -\langle \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \boldsymbol{u} \,,\, a \rangle_{\bar{\Omega}_\mathrm{g}} \tag{2.30}
$$

3.
$$
\overset{\times}{\nabla}{}'_h \cdot \overset{\times}{\nabla}_h a = \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \overset{\times}{\nabla}{}^{\perp}_h a = \overset{\times}{\triangle}{}'_h a \qquad \text{on } \bar{\Omega}_\mathrm{g}; \tag{2.31}
$$

4. *If $a \in L^2(\bar{\Omega}_\mathrm{g}, \mathbb{R})$, then*
$$
\overset{\times}{\nabla}{}'_h \cdot \overset{\times}{\nabla}{}^{\perp}_h a = \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \overset{\times}{\nabla}_h a = 0 \qquad \text{on } \mathring{\Omega}_\mathrm{g}. \tag{2.32}
$$

In addition, if $a \in L^2_c(\bar{\Omega}_\mathrm{g}, \mathbb{R})$, then
$$
\overset{\times}{\nabla}{}'_h \cdot \overset{\times}{\nabla}{}^{\perp}_h a = \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \overset{\times}{\nabla}_h a = 0 \qquad \text{on } \bar{\Omega}_\mathrm{g}. \tag{2.33}
$$

In addition to Lemma 2.1, the reduced operators also provide a way of incorporating the no-slip, no-penetration conditions at the physical boundary. The resulting scheme for (1.1) is given by

$$
\begin{aligned}
\boldsymbol{u}_t + \bar{\omega}\boldsymbol{u}^{\perp} + \overset{\times}{\nabla}_h p &= \nu \overset{\times}{\nabla}{}^{\perp}_h \omega + \boldsymbol{f} && \text{on } \Omega_\mathrm{c} \\
\omega &= \overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \boldsymbol{u} && \text{on } \bar{\Omega}_\mathrm{g} \\
\overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u} &= 0 && \text{on } \bar{\Omega}_\mathrm{g}
\end{aligned} \tag{2.34}
$$

The reduced divergence operator in the third equation of (2.34) has implicitly incorporated the no-penetration condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ in a natural way. On the other hand, the reduced curl operator in the second equation of (2.34) has implicitly incorporated the no-slip condition $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ_g. This can be interpreted as an implicit form of local vorticity boundary condition.

3. Error Estimate for the Generalized MAC Scheme. We now proceed with our main result, the second order error estimate for the generalized MAC scheme. Rigorous 2nd order error estimate for the classical MAC scheme and some variants were first obtained in [HW] and [W1]. The method used in [HW, W1] is a combination of energy estimate and high order Strang's expansion. Here we propose an alternative proof and apply it to our scheme. In addition to extending the analysis to curvilinear domains, our method differs from [HW, W1] in several aspects. The first new component in our analysis is to utilize the stream function and combine it with the discrete identity (2.33) in our analysis. As a result, the regularity requirement on the exact solution becomes transparent and less stringent. Secondly, our pressure error analysis is established via uniform inf-sup (LBB) estimate, which is of independent importance and has potential applications in other areas such as computational elasticity and computational electromagnetics. The verification of the inf-sup condition is quite technical and is left in the Appendix.

3.1. Basic Error Estimate. Our first main result, 2nd order error estimate for the velocity field, is obtained from basic energy estimate.

THEOREM 1. *Assume the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* C^4 *bijection from* $[0,1] \times S^1$ *to* $\bar{\Omega} \subset \mathbb{R}^2$. *Let* $\boldsymbol{u}^e \in L^2(0,T;C^4(\bar{\Omega})) \cap H^1(0,T;C^2(\bar{\Omega}))$, $p^e \in L^2(0,T;C^3(\bar{\Omega}))$ *be an exact solution of* (1.1), *and* $\boldsymbol{u}^h$, ω^h, p^h *the numerical solution of* (2.34) *with initial velocity* $\boldsymbol{u}_0^h$ *satisfying* $\overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u}_0^h = 0$. *Then we have*

$$\max_{[0,T]} \|\boldsymbol{u}^h - \boldsymbol{u}^e\|^2_{\Omega_c} + \nu \int_0^T \|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \le K_1(\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_c} + \overset{\times}{h}{}^4), \tag{3.1}$$

$$\int_0^T \|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \le K_2\big(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_c} + \overset{\times}{h}{}^2\big). \tag{3.2}$$

where K_1, K_2 *are constants depending on* T, ν, $\|\boldsymbol{u}^e\|_{L^2(0,T;C^4(\bar{\Omega}))}$, $\|\boldsymbol{u}^e\|_{H^1(0,T;C^2(\bar{\Omega}))}$, $\|p^e\|_{L^2(0,T;C^3(\bar{\Omega}))}$, *but not on* $\overset{\times}{h}$.

A crucial part of Theorem 1 is to utilize the exact stream function ψ^e to construct a divergence free approximate solution of the form

$$\boldsymbol{u}^a(t,\cdot) = \overset{\times}{\nabla}{}^{\perp}_h(\psi^e + \overset{\times}{h}{}^2\varphi) \in L^2(\Omega_c, \mathbb{R}^2) \tag{3.3}$$

such that both $\boldsymbol{u}^a - \boldsymbol{u}^e$ and $\boldsymbol{u}^h - \boldsymbol{u}^a$ are $O(h^2)$. The latter requires in addition that $\overset{\times}{\nabla}{}^{\perp\prime}_h \cdot \boldsymbol{u}^a - \omega^e = \overset{\times}{\triangle}{}'_h(\psi^e + \overset{\times}{h}{}^2\varphi) - \omega^e = O(h^2)$. Here φ is a correction to be determined from the following Lemma.

LEMMA 3.1. *If the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* C^4 *bijection from* $[0,1] \times S^1$ *to* $\bar{\Omega} \subset \mathbb{R}^2$ *and* $\psi \in C^4(\bar{\Omega})$, *then*

$$\begin{aligned}
\overset{\times}{\triangle}{}'_h\psi\big(\boldsymbol{x}(0,\xi^2)\big) &= \Big(\frac{2\sqrt{2}}{\overset{\times}{h}}(g^{11}\partial_1\psi + g^{12}\partial_2\psi) + \triangle\psi\Big)\big(\boldsymbol{x}(0,\xi^2)\big) + P_L(\psi)(\xi^2)\overset{\times}{h} + Q^h_L(\psi)(\xi^2)\overset{\times}{h}{}^2, \\
\overset{\times}{\triangle}{}'_h\psi\big(\boldsymbol{x}(1,\xi^2)\big) &= \Big(\frac{-2\sqrt{2}}{\overset{\times}{h}}(g^{11}\partial_1\psi + g^{12}\partial_2\psi) + \triangle\psi\Big)\big(\boldsymbol{x}(1,\xi^2)\big) + P_R(\psi)(\xi^2)\overset{\times}{h} + Q^h_R(\psi)(\xi^2)\overset{\times}{h}{}^2,
\end{aligned} \tag{3.4}$$

where

$$\|P_{L,R}(\psi)\|_{C^1(S^1)} + \|Q^h_{L,R}(\psi)\|_{C^0(S^1)} \le C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\|\psi\|_{C^4(\bar{\Omega})}. \tag{3.5}$$

Here in (3.5) and the rest of the paper, we denote by $C\big[\cdots\big]$ a positive constant that depends on the arguments inside the bracket.

Proof. We apply the reduced Laplacian (2.25) to ψ on $\boldsymbol{x}(0,\xi^2)$ and split it into two parts:

$$\overset{\times}{\triangle}{}'_h\psi\big(\boldsymbol{x}(0,\xi^2)\big) = \frac{2}{\sqrt{\overset{\times}{g}(0,\xi^2)}}\big(I_{L,+}(0,\xi^2) + I_{L,-}(0,\xi^2)\big) \tag{3.6}$$

where $I_{L,\pm}(0,\xi^2)$ are the terms associated with $\overset{\times}{q}^{\alpha\beta}(\frac{h}{2},\xi^2\pm\frac{h}{2}) = \sqrt{\overset{\times}{g}}\overset{\times}{g}^{\alpha\beta}(\frac{h}{2},\xi^2\pm\frac{h}{2})$, respectively. More precisely,

$$
\begin{aligned}
I_{L,+}(0,\xi^2) \triangleq\ & \frac{1}{\overset{\times}{h}}\Big(\overset{\times}{q}^{11}\overset{\times}{D}_1\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{D}_2\tilde{\psi}\Big)(\tfrac{h}{2},\xi^2+\tfrac{h}{2})\\
=\ & \frac{1}{\overset{\times}{h}}\overset{\times}{q}^{11}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})\Big(\overset{\times}{\partial}_1\tilde{\psi}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})+\frac{\overset{\times}{h}^2}{24}\overset{\times}{\partial}{}_1^3\tilde{\psi}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})+K^h_{L,1,+}(\psi)(\xi^2)\overset{\times}{h}^3\Big)\\
&+\frac{1}{\overset{\times}{h}}\overset{\times}{q}^{12}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})\Big(\overset{\times}{\partial}_2\tilde{\psi}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})+\frac{\overset{\times}{h}^2}{24}\overset{\times}{\partial}{}_2^3\tilde{\psi}(\tfrac{h}{2},\xi^2+\tfrac{h}{2})+K^h_{L,2,+}(\psi)(\xi^2)\overset{\times}{h}^3\Big)
\end{aligned}
\tag{3.7}
$$

where $\tilde{\psi}=\psi\circ\boldsymbol{x}$ and the remainder terms satisfy $|K^h_{L,\ell,+}(\psi)(\xi^2)|\le C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}\big]\|\psi\|_{C^4(\bar{\Omega})}$, and we have assumed that analytic metric tensors have been adopted in the discretization. The analysis below applies to numerical metric tensors without difficulty.

Expand (3.7) around $(0,\xi^2)$ and apply (2.2), we get

$$
\begin{aligned}
I_{L,+}(0,\xi^2) =&\frac{1}{\overset{\times}{h}}\big(\overset{\times}{q}^{11}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)+\frac{1}{2}\overset{\times}{\partial}_1\big(\overset{\times}{q}^{11}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)\\
&+\tilde{P}_{L,+}(\psi)(\xi^2)\overset{\times}{h}+\tilde{Q}^h_{L,+}(\psi)(\xi^2)\overset{\times}{h}^2
\end{aligned}
\tag{3.8}
$$

where $|\tilde{Q}^h_{L,+}(\psi)(\xi^2)|\le C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)},\|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\|\psi\|_{C^4(\bar{\Omega})}$ and

$$
\tilde{P}_{L,+}(\psi)(\xi^2)\triangleq\frac{1}{8}\overset{\times}{\partial}{}_1^2\big(\overset{\times}{q}^{11}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)+\frac{1}{24}\big(\overset{\times}{q}^{11}\overset{\times}{\partial}{}_1^3\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{\partial}{}_2^3\tilde{\psi}\big)(0,\xi^2).
\tag{3.9}
$$

Similarly,

$$
\begin{aligned}
I_{L,-}(0,\xi^2)\triangleq&\frac{-1}{\overset{\times}{h}}\Big(\overset{\times}{q}^{21}\overset{\times}{D}_1\tilde{\psi}+\overset{\times}{q}^{22}\overset{\times}{D}_2\tilde{\psi}\Big)(\tfrac{h}{2},\xi^2-\tfrac{h}{2})\\
=&-\frac{1}{\overset{\times}{h}}\big(\overset{\times}{q}^{21}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{22}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)+\frac{1}{2}\overset{\times}{\partial}_2\big(\overset{\times}{q}^{21}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{22}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)\\
&-\tilde{P}_{L,-}(\psi)(\xi^2)\overset{\times}{h}+\tilde{Q}^h_{L,-}(\psi)(\xi^2)\overset{\times}{h}^2
\end{aligned}
\tag{3.10}
$$

where $|\tilde{Q}^h_{L,-}(\psi)(\xi^2)|\le C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)},\|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\|\psi\|_{C^4(\bar{\Omega})}$ and

$$
\tilde{P}_{L,-}(\psi)(\xi^2)\triangleq\frac{1}{8}\overset{\times}{\partial}{}_2^2\big(\overset{\times}{q}^{21}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{22}\overset{\times}{\partial}_2\tilde{\psi}\big)(0,\xi^2)+\frac{1}{24}\big(\overset{\times}{q}^{21}\overset{\times}{\partial}{}_1^3\tilde{\psi}+\overset{\times}{q}^{22}\overset{\times}{\partial}{}_2^3\tilde{\psi}\big)(0,\xi^2).
\tag{3.11}
$$

From (2.1), (2.8) and the relation

$$
\overset{\times}{g}^{11}-2\overset{\times}{g}^{21}+\overset{\times}{g}^{22}=2g^{11},\qquad \overset{\times}{g}^{11}-\overset{\times}{g}^{22}=2g^{12},
\tag{3.12}
$$

we have

$$
\frac{1}{\overset{\times}{h}}\Big(\overset{\times}{q}^{11}\overset{\times}{\partial}_1\tilde{\psi}+\overset{\times}{q}^{12}\overset{\times}{\partial}_2\tilde{\psi}-\overset{\times}{q}^{21}\overset{\times}{\partial}_1\tilde{\psi}-\overset{\times}{q}^{22}\overset{\times}{\partial}_2\tilde{\psi}\Big)=\frac{\sqrt{2}\sqrt{\overset{\times}{g}}}{\overset{\times}{h}}\big(g^{11}\partial_1\psi+g^{12}\partial_2\psi\big).
\tag{3.13}
$$

As a consequence, (3.4) follows from (3.6), (3.8) and (3.10) with

$$P_L(\psi)(\xi^2) \triangleq \frac{2}{\sqrt{\overset{\times}{g}(0,\xi^2)}}\big(\tilde{P}_{L,+}(\psi)(\xi^2) - \tilde{P}_{L,-}(\psi)(\xi^2)\big),$$
$$Q_L^h(\psi)(\xi^2) \triangleq \frac{2}{\sqrt{\overset{\times}{g}(0,\xi^2)}}\big(\tilde{Q}_{L,+}^h(\psi)(\xi^2) + \tilde{Q}_{L,-}^h(\psi)(\xi^2)\big).$$

The estimate for P_R and Q_R^h are similar. ◻

From Lemma 3.1, it is clear that $|\omega^e(t,\cdot) - \omega^a(t,\cdot)| = O(h^2)$ provided φ satisfies

$$\begin{aligned} \varphi\big(t,\boldsymbol{x}(0,\xi^2)\big) &= 0, \qquad \partial_{\xi^1}\varphi\big(t,\boldsymbol{x}(0,\xi^2)\big) = \frac{-P_L(\psi^e)(t,\xi^2)}{2\sqrt{2}g^{11}(0,\xi^2)}, \\ \varphi\big(t,\boldsymbol{x}(1,\xi^2)\big) &= 0, \qquad \partial_{\xi^1}\varphi\big(t,\boldsymbol{x}(1,\xi^2)\big) = \frac{P_R(\psi^e)(t,\xi^2)}{2\sqrt{2}g^{11}(1,\xi^2)}. \end{aligned} \tag{3.14}$$

with $P_{L,R}(\psi^e)$ obtained by applying Lemma 3.1 to the exact stream function $\psi^e(t,\cdot)$.

Such a correction φ could be constructed by combining direct tensor products of $f_{L,R}$ with proper cutoff functions in ξ^1. However, for regularity consideration, we will elaborate further by mollifying f_L and f_R in the ξ^2 direction, with support of the mollifier proportional to the distance to Γ. The singularity induced by vanishing support of the mollifier near the boundary can be compensated by the condition $\varphi = 0$ on Γ. More precisely, we have the following Lemma:

Lemma 3.2. *Given f_L and $f_R \in C^1(S^1)$, there exists a function $\tilde{\varphi} \in C^2([0,1]\times S^1)$ such that*

$$\tilde{\varphi}(0,\xi^2) = 0, \qquad \tilde{\varphi}(1,\xi^2) = 0, \tag{3.15}$$

$$\partial_1\tilde{\varphi}(0,\xi^2) = f_L(\xi^2), \qquad \partial_1\tilde{\varphi}(1,\xi^2) = f_R(\xi^2), \tag{3.16}$$

and

$$\|\tilde{\varphi}\|_{C^2([0,1]\times S^1)} \le C\big(\|f_L\|_{C^1(S^1)} + \|f_R\|_{C^1(S^1)}\big). \tag{3.17}$$

Proof. Let $\eta(\cdot): S^1 \mapsto \mathbb{R}$ be a standard mollifier with compact support $[-\delta,\delta] \subset S^1$ and total mass 1,

$$\int_{S^1} \eta(\xi^2)\,d\xi^2 = 1, \qquad \eta^\epsilon(\xi^2) \triangleq \frac{1}{\epsilon}\eta(\frac{\xi^2}{\epsilon}).$$

We define

$$\tilde{\varphi}_L(\xi^1,\xi^2) = \xi^1\left(\eta^{\xi^1} * f_L\right)(\xi^2) = \int_{S^1} \eta(\frac{\xi^2-\lambda}{\xi^1}) f_L(\lambda)\,d\lambda, \tag{3.18}$$

$$\tilde{\varphi}_R(\xi^1,\xi^2) = (1-\xi^1)\big(\eta^{1-\xi^1} * f_R\big)(\xi^2) = \int_{S^1} \eta(\frac{\xi^2-\lambda}{1-\xi^1}) f_R(\lambda)\,d\lambda, \tag{3.19}$$

and

$$\tilde{\varphi}(\xi^1,\xi^2) = \tilde{\varphi}_L(\xi^1,\xi^2)\Theta(\xi^1) - \tilde{\varphi}_R(\xi^1,\xi^2)\Theta(1-\xi^1), \tag{3.20}$$

where Θ is a smooth cutoff function satisfying

$$\Theta(\xi^1) = \begin{cases} 1, & 0 \le \xi^1 \le \frac{1}{3} \\ 0, & \frac{2}{3} \le \xi^1 \le 1 \\ \text{smoothly connected on } \frac{1}{3} \le \xi^1 \le \frac{2}{3} \end{cases} \tag{3.21}$$

It is easy to see that $\tilde{\varphi}$ satisfies (3.15). To show that $\tilde{\varphi}$ satisfies (3.16), we note that

$$\partial_{\xi^1}(\xi^1 \eta^{\xi^1}(\xi^2)) = \partial_{\xi^1} \eta\big(\frac{\xi^2}{\xi^1}\big) = \frac{-\xi^2}{(\xi^1)^2}\eta'\big(\frac{\xi^2}{\xi^1}\big) = \frac{1}{\xi^1}\zeta\big(\frac{\xi^2}{\xi^1}\big) = \zeta^{\xi^1}(\xi^2), \tag{3.22}$$

where $\zeta(\xi^2) \triangleq -\xi^2\eta'(\xi^2)$, is another mollifier with total mass 1 and $\zeta^\epsilon(\xi^2) \triangleq \frac{1}{\epsilon}\zeta(\frac{\xi^2}{\epsilon})$. Therefore

$$\lim_{\xi^1\to 0^+} \partial_{\xi^1}\tilde{\varphi}_L(\xi^1,\xi^2) = \lim_{\xi^1\to 0^+}\int_{S^1} \partial_{\xi^1}(\xi^1\eta^{\xi^1})(\xi^2-\lambda) f_L(\lambda)\,d\lambda = \lim_{\xi^1\to 0^+}\int_{S^1} \zeta^{\xi^1}(\xi^2-\lambda)f_L(\lambda)\,d\lambda = f_L(\xi^2). \tag{3.23}$$

Similarly,

$$\lim_{\xi^1\to 1^-} \partial_{\xi^1}\tilde{\varphi}_R(\xi^1,\xi^2) = -f_R(\xi^2). \tag{3.24}$$

To estimate the C^2-norm of $\tilde{\varphi}(\xi^1,\xi^2)$, we first observe that

$$\begin{aligned} \big|\partial^2_{\xi^2}\tilde{\varphi}_L(\xi^1,\xi^2)\big| &= \big|\int_{S^1}\partial^2_{\xi^2}\,\eta(\frac{\xi^2-\lambda}{\xi^1})f_L(\lambda)\,d\lambda\big| = \big|\int_{S^1}\partial_\lambda\partial_{\xi^2}\,\eta(\frac{\xi^2-\lambda}{\xi^1})f_L(\lambda)\,d\lambda\big| \\ &= \big|\int_{S^1}\partial_{\xi^2}\,\eta(\frac{\xi^2-\lambda}{\xi^1})\partial_\lambda f_L(\lambda)\,d\lambda\big| \le \|f_L\|_{C^1(S^1)}\|\eta'\|_{L^1(S^1)}. \end{aligned} \tag{3.25}$$

Secondly

$$\begin{aligned} \partial_{\xi^1}\zeta^{\xi^1}(\xi^2-\lambda) = \partial_{\xi^1}\big(\frac{1}{\xi^1}\zeta(\frac{\xi^2-\lambda}{\xi^1})\big) &= \frac{-1}{\xi^1}\big(\frac{1}{\xi^1}\zeta(\frac{\xi^2-\lambda}{\xi^1}) + \frac{\xi^2-\lambda}{(\xi^1)^2}\zeta'(\frac{\xi^2-\lambda}{\xi^1})\big) \\ &= \frac{1}{\xi^1}\partial_\lambda\big(\frac{\xi^2-\lambda}{\xi^1}\zeta(\frac{\xi^2-\lambda}{\xi^1})\big) = \partial_\lambda Z^{\xi^1}(\xi^2-\lambda) \end{aligned} \tag{3.26}$$

where

$$Z(\xi^2) \triangleq \xi^2\zeta(\xi^2), \qquad Z^\epsilon(\xi^2) \triangleq \frac{1}{\epsilon}Z\big(\frac{\xi^2}{\epsilon}\big) \tag{3.27}$$

From (3.22) and (3.26), we have

$$\begin{aligned} |\partial^2_{\xi^1}\tilde{\varphi}_L(\xi^1,\xi^2)| &= \big|\int_{S^1}\partial_{\xi^1}\zeta^{\xi^1}(\xi^2-\lambda)f_L(\lambda)\,d\lambda\big| = \big|\int_{S^1}\partial_\lambda Z^{\xi^1}(\xi^2-\lambda)f_L(\lambda)\,d\lambda\big| \\ &= \big|\int_{S^1} Z^{\xi^1}(\xi^2-\lambda)\partial_\lambda f_L(\lambda)\,d\lambda\big| \le \|f_L\|_{C^1(S^1)}\|Z\|_{L^1(S^1)}. \end{aligned} \tag{3.28}$$

Similar calculation leads to

$$|\partial_{\xi^1}\tilde{\varphi}_L(\xi^1,\xi^2)| \le \|f_L\|_{C^0(S^1)}, \tag{3.29}$$

$$|\partial_{\xi^2}\tilde{\varphi}_L(\xi^1,\xi^2)| \le \|f_L\|_{C^0(S^1)}\|\eta'\|_{L^1(S^1)}, \tag{3.30}$$

$$|\partial_{\xi^1}\partial_{\xi^2}\tilde{\varphi}_L(\xi^1,\xi^2)| \le \|f_L\|_{C^1(S^1)}. \tag{3.31}$$

The estimate of $\tilde{\varphi}_R(\xi^1,\xi^2)$ is also similar. Therefore (3.16) follows. $\square$

Proof of Theorem 1. Let $\tilde{\varphi}$ be given by Lemma 3.2 with

$$f_L(t,\xi^2) \triangleq \frac{-P_L(\psi^e)(t,\xi^2)}{2\sqrt{2}g^{11}(0,\xi^2)}, \qquad f_R(t,\xi^2) \triangleq \frac{P_R(\psi^e)(t,\xi^2)}{2\sqrt{2}g^{11}(1,\xi^2)}. \tag{3.32}$$

and $\varphi = \tilde{\varphi} \circ \boldsymbol{x}^{-1}$. It follows from (3.5) that

$$\begin{aligned}\|\varphi(t,\cdot)\|_{C^2(\bar{\Omega})} \leq & C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\big(\|P_L(\psi^e)\|_{C^1(S^1)} + \|P_R(\psi^e)\|_{C^1(S^1)}\big)\\ \leq & C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\|\psi^e(t,\cdot)\|_{C^4(\bar{\Omega})}.\end{aligned} \tag{3.33}$$

From the construction of φ, it is also clear that

$$\begin{aligned}\|\partial_t\varphi(t,\cdot)\|_{C^2(\bar{\Omega})} \leq & C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\big(\|\partial_t P_L(\psi^e)\|_{C^1(S^1)} + \|\partial_t P_R(\psi^e)\|_{C^1(S^1)}\big)\\ \leq & C\big[\|\boldsymbol{x}\|_{C^4([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^4(\bar{\Omega})}\big]\|\partial_t\psi^e(t,\cdot)\|_{C^4(\bar{\Omega})}.\end{aligned} \tag{3.34}$$

Now we define

$$\boldsymbol{u}^a(t,\cdot) \triangleq \overset{\times}{\nabla}{}_h^{\perp}(\psi^e + \overset{\times}{h}{}^2\varphi) \in L^2(\Omega_{\rm c},\mathbb{R}^2) \tag{3.35}$$

The corresponding approximate vorticity is given by

$$\omega^a(t,\cdot) \triangleq \overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u}^a = \overset{\times}{\triangle}{}_h'(\psi^e + \overset{\times}{h}{}^2\varphi) \in L^2(\bar{\Omega}_{\rm g},\mathbb{R}). \tag{3.36}$$

It follows that

$$\omega^a = \begin{cases}\omega^e + \Big(P_{L,R}(\psi^e) \pm 2\sqrt{2}g^{11}\partial_1\varphi\Big)\overset{\times}{h} + \Big(R_4^h(\psi^e) + R_2^h(\varphi)\Big)\overset{\times}{h}{}^2 & \text{on } \Gamma_{\rm g}\\ \omega^e + \Big(\mathring{R}_4^h(\psi^e) + \mathring{R}_2^h(\varphi)\Big)\overset{\times}{h}{}^2 & \text{on } \mathring{\Omega}_{\rm g}\end{cases} \tag{3.37}$$

where the remainder terms satisfy $\|R_k^h(\cdot)\|_{C^0(\bar{\Omega})}, \|\mathring{R}_k^h(\cdot)\|_{C^0(\bar{\Omega})} \leq C\big[\|\boldsymbol{x}\|_{C^k([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^k(\bar{\Omega})}\big]\|\cdot\|_{C^k(\bar{\Omega})}$.

For brevity, we shall omit the dependence of C on the mapping $\boldsymbol{x}$ from now on. From (3.35), (3.36), (3.37), (3.14) and (2.33), we conclude that

$$|\boldsymbol{u}^a(t,\cdot) - \boldsymbol{u}^e(t,\cdot)| \leq C_1\big[\|\psi^e(t,\cdot)\|_{C^3(\bar{\Omega})}, \|\varphi(t,\cdot)\|_{C^1(\bar{\Omega})}\big]\overset{\times}{h}{}^2 = C_1\big[\|\boldsymbol{u}^e(t,\cdot)\|_{C^2(\bar{\Omega})}\big]\overset{\times}{h}{}^2, \tag{3.38}$$

$$|\partial_t\boldsymbol{u}^a(t,\cdot) - \partial_t\boldsymbol{u}^e(t,\cdot)| \leq C_2\big[\|\partial_t\psi^e(t,\cdot)\|_{C^3(\bar{\Omega})}, \|\partial_t\varphi(t,\cdot)\|_{C^1(\bar{\Omega})}\big]\overset{\times}{h}{}^2 = C_2\big[\|\partial_t\boldsymbol{u}^e(t,\cdot)\|_{C^2(\bar{\Omega})}\big]\overset{\times}{h}{}^2, \tag{3.39}$$

$$|\omega^a(t,\cdot) - \omega^e(t,\cdot)| \leq C_3\big[\|\psi^e(t,\cdot)\|_{C^4(\bar{\Omega})}, \|\varphi(t,\cdot)\|_{C^2(\bar{\Omega})}\big]\overset{\times}{h}{}^2 = C_3\big[\|\boldsymbol{u}^e(t,\cdot)\|_{C^3(\bar{\Omega})}\big]\overset{\times}{h}{}^2 \tag{3.40}$$

and

$$\overset{\times}{\nabla}{}_h'\cdot\boldsymbol{u}^h(t,\cdot) = \overset{\times}{\nabla}{}_h'\cdot\boldsymbol{u}^a(t,\cdot) = 0. \tag{3.41}$$

We can now write

$$\partial_t\boldsymbol{u}^a + \bar{\omega}^e\boldsymbol{u}^{a\perp} + \overset{\times}{\nabla}_h p^e = \nu\overset{\times}{\nabla}{}_h^{\perp}\omega^e + \mathscr{E} + \boldsymbol{f} \qquad \text{on } \Omega_{\rm c}, \tag{3.42}$$

where $\mathscr{E}$ is the local truncation error:

$$\mathscr{E} = \partial_t(\boldsymbol{u}^a - \boldsymbol{u}^e) + (\bar{\omega}^e \boldsymbol{u}^{a\perp} - \omega^e \boldsymbol{u}^{e\perp}) + (\overset{\times}{\nabla}_h - \nabla)p^e + \nu(\nabla^\perp - \overset{\times}{\nabla}{}_h^\perp)\omega^e. \tag{3.43}$$

From (3.38, 3.39) and

$$\bar{\omega}^e \boldsymbol{u}^{a\perp} - \omega^e (\boldsymbol{u}^e)^\perp = \bar{\omega}^e(\boldsymbol{u}^a - \boldsymbol{u}^e)^\perp + (\bar{\omega}^e - \omega^e)(\boldsymbol{u}^e)^\perp, \tag{3.44}$$

it is easy to see that

$$\int_0^T \|\mathscr{E}\|^2_{\Omega_c} \le C_4\big[\nu, \|\boldsymbol{u}^e\|_{H^1(0,T;C^2(\bar{\Omega}))}, \|\boldsymbol{u}^e\|_{L^2(0,T;C^4(\bar{\Omega}))}, \|p^e\|_{L^2(0,T;C^3(\bar{\Omega}))}\big]\overset{\times}{h}{}^4. \tag{3.45}$$

We now proceed to derive an error equation. Define $\boldsymbol{\varepsilon}(t,\cdot) = \boldsymbol{u}^h(t,\cdot) - \boldsymbol{u}^a(t,\cdot)$, we have

$$\partial_t \boldsymbol{\varepsilon} + (\bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}) + \overset{\times}{\nabla}_h(p^h - p^e) = \nu \overset{\times}{\nabla}{}_h^\perp(\omega^h - \omega^e) - \mathscr{E}. \tag{3.46}$$

From (2.29) and (3.41), we have $\langle\, \boldsymbol{\varepsilon}\,,\, \overset{\times}{\nabla}_h(p^h - p^e)\,\rangle_{\Omega_c} = -\langle\, \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{\varepsilon}\,,\, p^h - p^e\,\rangle_{\bar{\Omega}_g} = 0$, and thus

$$\langle\, \boldsymbol{\varepsilon}\,,\, \partial_t\boldsymbol{\varepsilon}\,\rangle_{\Omega_c} + \langle\, \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c} = \nu\langle\, \boldsymbol{\varepsilon}\,,\, \overset{\times}{\nabla}{}_h^\perp(\omega^h - \omega^e)\,\rangle_{\Omega_c} - \langle\, \boldsymbol{\varepsilon}\,,\, \mathscr{E}\,\rangle_{\Omega_c}.$$

From (2.30),

$$\begin{aligned}
\nu\langle\, \boldsymbol{\varepsilon}\,,\, \overset{\times}{\nabla}{}_h^\perp(\omega^h - \omega^e)\,\rangle_{\Omega_c} &= -\nu\langle\, \overset{\times}{\nabla}{}_h^{\perp\prime} \cdot (\boldsymbol{u}^h - \boldsymbol{u}^a)\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_g} \\
&= -\nu\langle\, \omega^h - \omega^a\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_g} \\
&= -\nu\langle\, \omega^h - \omega^e\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_g} - \nu\langle\, \omega^e - \omega^a\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_g} \\
&\le -\frac{\nu}{2}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} + \frac{\nu}{2}\|\omega^e - \omega^a\|^2_{\bar{\Omega}_g}.
\end{aligned}$$

Since $\boldsymbol{\varepsilon} = \boldsymbol{u}^h - \boldsymbol{u}^a$ is pointwise perpendicular to $\bar{\omega}^h(\boldsymbol{u}^h - \boldsymbol{u}^a)^\perp$, we have

$$\langle\, \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c} = \langle\, \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h(\boldsymbol{u}^h - \boldsymbol{u}^a)^\perp + (\bar{\omega}^h - \bar{\omega}^e)\boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c} = \langle\, \boldsymbol{\varepsilon}\,,\, (\bar{\omega}^h - \bar{\omega}^e)\boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c}.$$

Thus from Jensen's inequality,

$$\big|\langle\, \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c}\big| \le \frac{1}{\nu}\|\boldsymbol{u}^a\|^2_{L^\infty(\Omega_c)}\|\boldsymbol{\varepsilon}\|^2_{\Omega_c} + \frac{\nu}{4}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \le \frac{1}{\nu}\|\boldsymbol{u}^a\|^2_{L^\infty(\Omega_c)}\|\boldsymbol{\varepsilon}\|^2_{\Omega_c} + \frac{\nu}{4}\|\bar{\omega}^h - \bar{\omega}^e\|^2_{\Omega_c}$$

In summary, we have shown that

$$\frac{1}{2}\partial_t\|\boldsymbol{\varepsilon}\|^2_{\Omega_c} + \frac{\nu}{4}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \le \Big(\frac{1}{2} + \frac{1}{\nu}\|\boldsymbol{u}^a\|^2_{L^\infty(\Omega_c)}\Big)\|\boldsymbol{\varepsilon}\|^2_{\Omega_c} + \frac{\nu}{2}\|\omega^e - \omega^a\|^2_{\bar{\Omega}_g} + \frac{1}{2}\|\mathscr{E}\|^2_{\Omega_c}. \tag{3.47}$$

In view of (3.40), (3.45), and Gronwall's inequality, we have

$$\max_{0\le t\le T}\|\boldsymbol{u}^h - \boldsymbol{u}^a\|^2_{\Omega_c} + \nu\int_0^T \|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \le K_1(\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c} + \overset{\times}{h}{}^4). \tag{3.48}$$

The estimate (3.1) follows in view of (3.38).

Next, we proceed to show the preliminary pressure error estimate (3.2). Take the inner product with $\overset{\times}{\nabla}_h(p^h - p^e)$ on both sides of (3.46), then apply (2.29) and (3.41), we obtain

$$\begin{aligned}&\langle \overset{\times}{\nabla}_h(p^h - p^e)\, , \, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp} \rangle_{\Omega_c} + \|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \\ =&\nu \langle \overset{\times}{\nabla}_h(p^h - p^e)\, , \, \overset{\times}{\nabla}{}^{\perp}_h(\omega^h - \omega^e) \rangle_{\Omega_c} - \langle \overset{\times}{\nabla}_h(p^h - p^e)\, , \, \mathscr{E} \rangle_{\Omega_c}.\end{aligned} \tag{3.49}$$

Thus

$$\frac{1}{4}\|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \le \nu^2 \|\overset{\times}{\nabla}{}^{\perp}_h(\omega^h - \omega^e)\|^2_{\Omega_c} + \|\bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\|^2_{\Omega_c} + \|\mathscr{E}\|^2_{\Omega_c}. \tag{3.50}$$

Since

$$\|\overset{\times}{\nabla}{}^{\perp}_h(\omega^h - \omega^e)\|^2_{\Omega_c} \le \frac{8}{\overset{\times}{h}{}^2}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g}$$

and

$$\|\bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\|^2_{\Omega_c} \le 2\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_c)}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} + 2\|\omega^e\|^2_{C^0(\bar{\Omega})}\|\boldsymbol{u}^h - \boldsymbol{u}^a\|^2_{\Omega_c}, \tag{3.51}$$

it follows that

$$\begin{aligned}\frac{1}{4}\int_0^T \|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \le&\Big(\frac{8\nu^2}{\overset{\times}{h}{}^2} + 2\max_{[0,T]}\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_c)}\Big)\int_0^T \|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \\ &+ 2\max_{[0,T]}\|\omega^e\|^2_{C^0(\bar{\Omega})}\int_0^T \|\boldsymbol{u}^h - \boldsymbol{u}^a\|^2_{\Omega_c} + \int_0^T \|\mathscr{E}\|^2_{\Omega_c}.\end{aligned}$$

Note that

$$\begin{aligned}\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_c)} \le& 2\|\boldsymbol{u}^h - \boldsymbol{u}^e\|^2_{L^\infty(\Omega_c)} + 2\|\boldsymbol{u}^e\|^2_{C^0(\bar{\Omega})} \\ \le& \frac{2}{\overset{\times}{h}{}^2 \min\limits_{\Omega_c}\sqrt{g}_h}\|\boldsymbol{u}^h - \boldsymbol{u}^e\|^2_{\Omega_c} + 2\|\boldsymbol{u}^e\|^2_{C^0(\bar{\Omega})} \\ \le& \frac{2K_1}{\min\limits_{\Omega_c}\sqrt{g}_h}(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c} + \overset{\times}{h}{}^2) + 2\|\boldsymbol{u}^e\|^2_{C^0(\bar{\Omega})}.\end{aligned} \tag{3.52}$$

Hence

$$\max_{[0,T]}\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_c)} \le C_5(1 + \overset{\times}{h}{}^{-2}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c}), \tag{3.53}$$

where $C_5 = C_5\big[T, \nu, \|\boldsymbol{u}^e\|_{L^2(0,T;C^4(\bar{\Omega}))}, \|\boldsymbol{u}^e\|_{H^1(0,T;C^2(\bar{\Omega}))}, \|p^e\|_{L^2(0,T;C^3(\bar{\Omega}))}\big]$. Consequently,

$$\int_0^T \|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \le K_2'\big(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c}(1 + \|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c}) + \overset{\times}{h}{}^2\big) \le K_2\big(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c} + \overset{\times}{h}{}^2\big). \tag{3.54}$$

□

REMARK 1. *The error estimate* (3.1), (3.2) *is subject to appropriate approximation of the initial velocity field. This can be achieved, for example, by taking* $\boldsymbol{u}^h_0 = \boldsymbol{u}^a_0 = \boldsymbol{u}^a(0,\cdot)$ *with* $\boldsymbol{u}^a$ *defined by* (3.35). *In view of* (3.38), $\boldsymbol{u}^h_0 = \boldsymbol{u}^a_0$ *gives*

$$\max_{[0,T]}\|\boldsymbol{u}^h - \boldsymbol{u}^e\|^2_{\Omega_c} + \nu\int_0^T \|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} \le K_1\overset{\times}{h}{}^4, \tag{3.55}$$

$$\int_0^T \|\overset{\times}{\nabla}_h(p^h - p^e)\|^2_{\Omega_c} \le K_2\overset{\times}{h}{}^2. \tag{3.56}$$

Similarly, the refined estimates (3.58), (3.59) *in Theorem 3 also depend on the approximations of initial velocity* $\boldsymbol{u}_0^h - \boldsymbol{u}_0^e$ *and initial vorticity* $\omega_0^h - \omega_0^e = \overset{\times}{\nabla}{}_h^{\perp}(\boldsymbol{u}_0^h - \boldsymbol{u}_0^e)$. *The choice* $\boldsymbol{u}_0^h = \boldsymbol{u}_0^a$ *and consequently* $\omega_0^h = \overset{\times}{\nabla}{}_h^{\perp}\boldsymbol{u}_0^a$ *gives full second order accuracy in* (3.58), (3.59) *in view of* (3.38) *and* (3.40).

3.2. The LBB Condition and Refined Pressure Estimate. Since the pressure lacks an evolutionary equation, it is in general difficult to obtain optimal estimate from basic energy estimate alone. The pressure estimate (3.2) is only first order. To get full second order accuracy, we resort to the well known inf-sup condition (also known as div-stability condition or Ladyzhenskaya-Babuška-Brezzi (LBB) condition). Indeed, we have

THEOREM 2 (**LBB**). *If the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* $C^{1,1}$ *bijection between* $[0,1] \times S^1$ *and* $\bar{\Omega} \subset \mathbb{R}^2$, *then there exists a constant* $\beta > 0$ *such that*

$$\inf_{p \in L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2} \ \sup_{\boldsymbol{u} \in L^2(\Omega_c, \mathbb{R}^2)} \frac{\langle\, p \,,\, \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u} \,\rangle_{\bar{\Omega}_{\mathrm{g}}}}{\|p\|_{\bar{\Omega}_{\mathrm{g}}} \|\boldsymbol{u}\|_{\overset{\times}{H}{}_h^1}} \geq \beta \quad \text{uniformly in } h. \tag{3.57}$$

Here $\|\boldsymbol{u}\|_{\overset{\times}{H}{}_h^1} \triangleq (\|\boldsymbol{u}\|_{\Omega_c}^2 + \|\overset{\times}{\nabla}{}_h^{\perp\prime} \cdot \boldsymbol{u}\|_{\bar{\Omega}_{\mathrm{g}}}^2 + \|\overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u}\|_{\bar{\Omega}_{\mathrm{g}}}^2)^{\frac{1}{2}}$ is the natural norm associated with the discrete vector Laplacian.

The LBB condition arises naturally as a compatibility condition between the discrete velocity and pressure spaces in mixed finite element formulations. It is a fundamental research topic in finite element analysis for steady state computation, yet rarely discussed in finite difference setting or dynamical problems. The uniform estimate (3.57) is not only vital to the pressure error estimate presented here, but also directly affects the condition number and uniform bound of the solution operator for the Stokes's problem.

The verification of the LBB condition is quite complicated, especially for low order finite element methods. Denote by $\mathrm{DOF}(\boldsymbol{u})$ and $\mathrm{DOF}(p)$ the degree of freedom for $\boldsymbol{u}$ and p, respectively. It is clear that, the larger the ratio $\mathrm{DOF}(\boldsymbol{u})/\mathrm{DOF}(p)$ is, the more likely (3.57) is to hold and be verified.

To verify the LBB condition, a common approach is to reduce the problem to a patch of elements and conduct local analysis [BN1]. See also [St] for a related approach. This local argument has proved successful for higher order finite element methods with large $\mathrm{DOF}(\boldsymbol{u})/\mathrm{DOF}(p)$ ratio. When the ratio is low, the spatial compatibility plays a crucial role in establishing (3.57).

For example, the spaces (Q_l, P_{l-1}) are well known and widely used in mixed finite element formulation. Using the local argument, it can be shown that LBB condition holds for (Q_l, P_{l-1}) with $l \geq 2$ [GR]. However, this local argument does not apply to the lowest order scheme $Q_1 - P_0$ (bilinear in velocity components and piecewise constant in pressure), but only to some of its variants equipped with extra degrees of freedom in the velocity space. In fact, the $Q_1 - P_0$ element is known to violate the LBB condition with $\beta = O(h)$ [BN2].

In our case, GMAC is staggered and supported the same way as the $Q_1 - P_0$ element but only differs in the discretization of the vector Laplacian. Both of them have the minimal ratio $\mathrm{DOF}(\boldsymbol{u})/\mathrm{DOF}(p) = 2$ (3, if in 3D) among quadrilateral meshes. The similarity between GMAC and the $Q_1 - P_0$ element demonstrates the subtlety and difficulty of Theorem 2. It is also a common belief that staggered grids in some sense implies the inf-sup condition. The contrast between GMAC scheme and the $Q_1 - P_0$ element indicates the inf-sup estimate is subtler than plain staggeredness. Our analysis shows that it is closely related to the compatibility of the spatial discretization (2.29)-(2.32). See the Appendix for details.

With the inf-sup condition (3.57) established, the pressure error estimate can be improved to second order with only minor extra regularity requirement on the exact solution:

THEOREM 3. *Assume the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* C^4 *bijection from* $[0,1] \times S^1$ *to* $\bar{\Omega} \subset \mathbb{R}^2$, *and* $\boldsymbol{u}^e \in L^2(0,T;C^4(\bar{\Omega})) \cap H^1(0,T;C^3(\bar{\Omega}))$, $p^e \in L^2(0,T;C^3(\bar{\Omega}))$. *Then*

$$\nu \max_{[0,T]} \|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \int_0^T \|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)\|^2_{\Omega_{\mathrm{c}}} \leq K_3(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^4_{\Omega_{\mathrm{c}}} + \|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_{\mathrm{c}}} + \|\omega_0^h - \omega_0^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \overset{\times}{h}{}^4), \tag{3.58}$$

$$\int_0^T \|p^h - p^e\|^2_{\bar{\Omega}_{\mathrm{g}}} \leq K_4(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^4_{\Omega_{\mathrm{c}}} + \|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_{\mathrm{c}}} + \|\omega_0^h - \omega_0^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \overset{\times}{h}{}^4) \tag{3.59}$$

where K_3, K_4 *are constants that depend only on* T, ν, $\|\boldsymbol{u}^e\|_{H^1(0,T;C^3(\bar{\Omega}))}$,$\|\boldsymbol{u}^e\|_{L^2(0,T;C^4(\bar{\Omega}))}$, $\|p^e\|_{L^2(0,T;C^3(\bar{\Omega}))}$ *and independent of* h.

Proof. We start with the estimate (3.58). Take the inner product with $\partial_t \boldsymbol{\varepsilon}$ on both sides of (3.46) and apply (3.41), we get

$$\|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} + \langle\, \partial_t \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\,\rangle_{\Omega_{\mathrm{c}}} = -\nu \langle\, \partial_t(\omega^h - \omega^a)\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_{\mathrm{g}}} - \langle\, \partial_t \boldsymbol{\varepsilon}\,,\, \mathscr{E}\,\rangle_{\Omega_{\mathrm{c}}}.$$

Note that, from (3.51),

$$\begin{aligned}
&\frac{\nu}{2}\partial_t \|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} \\
\leq & |\langle\, \partial_t \boldsymbol{\varepsilon}\,,\, \bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\,\rangle_{\Omega_{\mathrm{c}}}| + \nu |\langle\, \partial_t(\omega^e - \omega^a)\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_{\mathrm{g}}}| + \frac{1}{4}\|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} + \|\mathscr{E}\|^2_{\Omega_{\mathrm{c}}} \\
\leq & \frac{1}{2}\|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} + \|\bar{\omega}^h \boldsymbol{u}^{h\perp} - \bar{\omega}^e \boldsymbol{u}^{a\perp}\|^2_{\Omega_{\mathrm{c}}} + \frac{\nu}{2}\|\partial_t(\omega^e - \omega^a)\|^2_{\bar{\Omega}_{\mathrm{g}}} + \frac{\nu}{2}\|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \|\mathscr{E}\|^2_{\Omega_{\mathrm{c}}} \\
\leq & \frac{1}{2}\|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} + \big(2\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_{\mathrm{c}})} + \frac{\nu}{2}\big)\|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + 2\|\omega^e\|^2_{C^0(\bar{\Omega})}\|\boldsymbol{u}^h - \boldsymbol{u}^a\|^2_{\Omega_{\mathrm{c}}} + \frac{\nu}{2}\|\partial_t(\omega^e - \omega^a)\|^2_{\bar{\Omega}_{\mathrm{g}}} + \|\mathscr{E}\|^2_{\Omega_{\mathrm{c}}}.
\end{aligned}$$

From (3.36), we have

$$\|\partial_t \omega^e(t,\cdot) - \partial_t \omega^a(t,\cdot)\|^2_{\bar{\Omega}_{\mathrm{g}}} \leq C_6\big[\|\partial_t \psi^e(t,\cdot)\|^2_{C^4(\bar{\Omega})}, \|\partial_t \varphi(t,\cdot)\|^2_{C^2(\bar{\Omega})}\big]\overset{\times}{h}{}^4 = C_6\big[\|\partial_t \boldsymbol{u}^e(t,\cdot)\|^2_{C^3(\bar{\Omega})}\big]\overset{\times}{h}{}^4. \tag{3.60}$$

Thus, in view of (3.48), we obtain

$$\begin{aligned}
&\nu \partial_t \|(\omega^h - \omega^e)(t,\cdot)\|^2_{\bar{\Omega}_{\mathrm{g}}} + \|\partial_t \boldsymbol{\varepsilon}(t,\cdot)\|^2_{\Omega_{\mathrm{c}}} \\
&\leq \big(4\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_{\mathrm{c}})} + \nu\big)\|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + C_7\big[\nu, K_1, \|\omega^e\|_{C^0(\bar{\Omega})}, C_6\big]\big(\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_{\mathrm{c}}} + \overset{\times}{h}{}^4\big) + \|\mathscr{E}\|^2_{\Omega_{\mathrm{c}}}.
\end{aligned} \tag{3.61}$$

Integrate (3.61) over $[0,T]$ and apply (3.1), (3.45) and (3.53), we have

$$\int_0^T \|\partial_t \boldsymbol{\varepsilon}\|^2_{\Omega_{\mathrm{c}}} + \nu \max_{[0,T]} \|\omega^h - \omega^e\|^2_{\bar{\Omega}_{\mathrm{g}}} \leq K_3(\overset{\times}{h}{}^{-2}\|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^4_{\Omega_{\mathrm{c}}} + \|\boldsymbol{u}_0^h - \boldsymbol{u}_0^e\|^2_{\Omega_{\mathrm{c}}} + \|\omega_0^h - \omega_0^e\|^2_{\bar{\Omega}_{\mathrm{g}}} + \overset{\times}{h}{}^4). \tag{3.62}$$

Thus (3.58) follows in view of (3.39).

We now proceed with the pressure error estimate (3.59). Firstly, both p^h and p^e are unique up to proper normalizations. The pressure norm in (3.59) should be understood as

$$\|p^h - p^e\|_{\bar{\Omega}_{\mathrm{g}}} \triangleq \min_{p^h - \tilde{p} \in \ker(\overset{\times}{\nabla}{}_h^{\perp})} \|\tilde{p} - p^e\|_{\bar{\Omega}_{\mathrm{g}}}$$

Without loss of generality, we assume that $\int_\Omega p^e dx = 0$ and $p^h - p^e \in L^2(\bar{\Omega}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$.

Since the assumption of Theorem 2 is weaker than that of Theorem 3, it follows that (3.57) holds and there exists $\boldsymbol{u} \in L^2(\Omega_{\mathrm{c}}, \mathbb{R}^2)$ such that

$$-\langle\, \overset{\times}{\nabla}_h (p^h - p^e)\,,\, \boldsymbol{u}\,\rangle_{\Omega_{\mathrm{c}}} = \langle\, p^h - p^e\,,\, \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u}\,\rangle_{\Omega_{\mathrm{c}}} \geq \beta \|(p^h - p^e)\|_{\bar{\Omega}_{\mathrm{g}}} \|\boldsymbol{u}\|_{\overset{\times}{H}{}^1_h}. \tag{3.63}$$

Now take the inner product with $-\boldsymbol{u}$ on both sides of (3.46), we get
(3.64)
$$\langle -\boldsymbol{u}\,,\, \partial_t\boldsymbol{\varepsilon}\,\rangle_{\Omega_c} + \langle -\boldsymbol{u}\,,\, \bar{\omega}^h\boldsymbol{u}^{h\perp} - \bar{\omega}^e\boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c} - \langle\, \overset{\times}{\nabla}_h(p^h - p^e)\,,\, \boldsymbol{u}\,\rangle_{\Omega_c} = \nu\langle -\boldsymbol{u}\,,\, \overset{\times}{\nabla}{}_h^{\perp}(\omega^h - \omega^e)\,\rangle_{\Omega_c} + \langle\, \boldsymbol{u}\,,\, \mathscr{E}\,\rangle_{\Omega_c}.$$

Therefore

$$\begin{aligned} &\beta\|(p^h - p^e)\|_{\bar{\Omega}_g}\|\boldsymbol{u}\|_{\overset{\times}{H}{}_h^1} \\ &\le |\langle\, \boldsymbol{u}\,,\, \partial_t\boldsymbol{\varepsilon}\,\rangle_{\Omega_c}| + |\langle\, \boldsymbol{u}\,,\, \bar{\omega}^h\boldsymbol{u}^{h\perp} - \bar{\omega}^e\boldsymbol{u}^{a\perp}\,\rangle_{\Omega_c}| + |\langle\, \boldsymbol{u}\,,\, \mathscr{E}\,\rangle_{\Omega_c}| + \nu|\langle\, \overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u}\,,\, \omega^h - \omega^e\,\rangle_{\bar{\Omega}_g}| \\ &\le \|\boldsymbol{u}\|_{\overset{\times}{H}{}_h^1}\Big(\|\partial_t\boldsymbol{\varepsilon}\|_{\Omega_c} + \|\bar{\omega}^h\boldsymbol{u}^{h\perp} - \bar{\omega}^e\boldsymbol{u}^{a\perp}\|_{\Omega_c} + \|\mathscr{E}\|_{\Omega_c} + \nu\|\omega^h - \omega^e\|_{\bar{\Omega}_g}\Big) \end{aligned} \tag{3.65}$$

Consequently, in view of (3.51), we obtain

$$\|p^h - p^e\|^2_{\bar{\Omega}_g} \le \frac{4}{\beta^2}\Big(\|\partial_t\boldsymbol{\varepsilon}\|^2_{\Omega_c} + \big(2\|\boldsymbol{u}^h\|^2_{L^\infty(\Omega_c)} + \nu^2\big)\|\omega^h - \omega^e\|^2_{\bar{\Omega}_g} + 2\|\omega^e\|^2_{C^0(\bar{\Omega})}\|\boldsymbol{u}^h - \boldsymbol{u}^a\|^2_{\Omega_c} + \|\mathscr{E}\|^2_{\Omega_c}\Big) \tag{3.66}$$

Now we integrate from 0 to T on both sides of (3.66) and apply (3.1, 3.39, 3.45, 3.53, 3.58). The refined pressure estimate (3.59) then follows. ∎

Similarly, one can give an $L^\infty(0,T;L^2(\bar{\Omega}_g))$ estimate for the pressure error with higher regularity requirement on the exact solution. We state the following Theorem without proof.

THEOREM 4. *Assume the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* C^4 *bijection from* $[0,1]\times S^1$ *to* $\bar{\Omega}\subset\mathbb{R}^2$, *and* $\boldsymbol{u}^e \in H^1(0,T;C^4(\bar{\Omega}))\cap H^2(0,T;C^2(\bar{\Omega}))$, $p^e\in H^1(0,T;C^3(\bar{\Omega}))$. *Then*

$$\max_{[0,T]}\|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)\|^2_{\Omega_c} + \nu\int_0^T\|\partial_t(\omega^h - \omega^e)\|^2_{\bar{\Omega}_g} \le K_5(\chi_1 + \|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)_0\|^2_{\Omega_c} + \overset{\times}{h}{}^4) \tag{3.67}$$

$$\max_{[0,T]}\|p^h - p^e\|^2_{\bar{\Omega}_g} \le K_6(\chi_1 + \|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)_0\|^2_{\Omega_c} + \overset{\times}{h}{}^4) \tag{3.68}$$

where

$$\chi_1 \triangleq \overset{\times}{h}{}^{-4}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^6_{\Omega_c} + \|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c} + \overset{\times}{h}{}^{-1}\|\omega^h_0 - \omega^e_0\|^3_{\bar{\Omega}_g} + \|\omega^h_0 - \omega^e_0\|^2_{\bar{\Omega}_g} \tag{3.69}$$

and K_5, K_6 *are constants that depend on* T, ν, $\|\boldsymbol{u}^e\|_{H^1(0,T;C^4(\bar{\Omega}))}$, $\|\boldsymbol{u}^e\|_{H^2(0,T;C^2(\bar{\Omega}))}$ *and* $\|p^e\|_{H^1(0,T;C^3(\bar{\Omega}))}$, *but not on* h.

The $\|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)_0\|^2_{\Omega_c}$ term in (3.67) and (3.68) result from Ladyzhenskaya type higher order energy estimate. An alternative expression in terms of $\|\overset{\times}{\nabla}{}_h^{\perp}(\omega^h_0 - \omega^e_0)\|^2_{\Omega_c}$ can be derived as follows.

Rewrite (3.46) at $t = 0$ as

$$\nu\overset{\times}{\nabla}{}_h^{\perp}(\omega^h_0 - \omega^e_0) - (\bar{\omega}^h_0\boldsymbol{u}^{h\perp}_0 - \bar{\omega}^e_0\boldsymbol{u}^{a\perp}_0) - \mathscr{E}_0 = \partial_t(\boldsymbol{u}^h - \boldsymbol{u}^a)_0 + \overset{\times}{\nabla}_h(p^h_0 - p^e_0). \tag{3.70}$$

Since

$$\langle\, \partial_t(\boldsymbol{u}^h - \boldsymbol{u}^a)_0\,,\, \overset{\times}{\nabla}_h(p^h_0 - p^e_0)\,\rangle_{\Omega_c} = 0, \tag{3.71}$$

it follows that (3.70) is an orthogonal decomposition for $\nu\overset{\times}{\nabla}{}_h^{\perp}(\omega^h_0 - \omega^e_0) - (\bar{\omega}^h_0\boldsymbol{u}^{h\perp}_0 - \bar{\omega}^e_0\boldsymbol{u}^{a\perp}_0) - \mathscr{E}_0$. Therefore

$$\|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^a)_0\|_{\Omega_c} \le \nu\|\overset{\times}{\nabla}{}_h^{\perp}(\omega^h_0 - \omega^e_0)\|_{\Omega_c} + \|\bar{\omega}^h_0\boldsymbol{u}^{h\perp}_0 - \bar{\omega}^e_0\boldsymbol{u}^{a\perp}_0\|_{\Omega_c} + \|\mathscr{E}_0\|_{\Omega_c}. \tag{3.72}$$

Moreover, from (3.43), (3.44), (3.38) and (3.39), we have

$$\|\mathscr{E}_0\|^2_{\Omega_c} \leq C_8\big[\|\boldsymbol{u}^e_0\|_{C^4(\bar\Omega)}, \|\partial_t \boldsymbol{u}^e_0\|_{C^2(\bar\Omega)}, \|p^e_0\|_{C^3(\bar\Omega)}\big]\overset{\times}{h}{}^4, \tag{3.73}$$

and

$$\|\bar\omega^h_0 \boldsymbol{u}^{h\perp}_0 - \bar\omega^e_0 \boldsymbol{u}^{a\perp}_0\|^2_{\Omega_c} \leq 2\|\boldsymbol{u}^h_0\|^2_{L^\infty(\Omega_c)}\|\omega^h_0 - \omega^e_0\|^2_{\bar\Omega_g} + 2\|\omega^e_0\|^2_{C^0(\bar\Omega)}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^a_0\|^2_{\Omega_c} \triangleq \chi_2, \tag{3.74}$$

with

$$\chi_2 \leq C\big[\|\boldsymbol{u}^e_0\|_{C^1(\bar\Omega)}\big]\big(\|\omega^h_0 - \omega^e_0\|^2_{\bar\Omega_g} + \|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c} + \overset{\times}{h}{}^{-2}\|\boldsymbol{u}^h_0 - \boldsymbol{u}^e_0\|^2_{\Omega_c}\|\omega^h_0 - \omega^e_0\|^2_{\bar\Omega_g}\big). \tag{3.75}$$

Therefore

$$\|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^a)_0\|^2_{\Omega_c} \leq \nu\|\overset{\times}{\nabla}{}^\perp_h(\omega^h_0 - \omega^e_0)\|^2_{\Omega_c} + \chi_2 + C_8\overset{\times}{h}{}^4 \leq C_9\big[\nu, C_2, C_8\big]\big(\|\overset{\times}{\nabla}{}^\perp_h(\omega^h_0 - \omega^e_0)\|^2_{\Omega_c} + \chi_2 + \overset{\times}{h}{}^4\big). \tag{3.76}$$

As a result, we can replace the $\|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)_0\|^2_{\Omega_c}$ term in (3.67) and (3.68) by $\|\overset{\times}{\nabla}{}^\perp_h(\omega^h_0 - \omega^e_0)\|^2_{\Omega_c} + \chi_2$, which is easier to analyze in practice.

For example, if the mapping $\boldsymbol{x}$ is a C^5 bijection and $\boldsymbol{u}^e_0 \in C^4(\bar\Omega)$ (and consequently $\psi^e_0 \in C^5(\bar\Omega)$), then following the derivation in the proof of Lemma 3.1, it is easy to see that

$$\|P_{L,R}(\psi^e_0)\|_{C^2(S^1)} \leq C\big[\|\boldsymbol{x}\|_{C^5([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^5(\bar\Omega)}\big]\|\psi^e_0\|_{C^5(\bar\Omega)} \tag{3.77}$$

and the construction (3.18)-(3.21) gives

$$\|\tilde\varphi\|_{C^3([0,1]\times S^1)} \leq C\big[\|\boldsymbol{x}\|_{C^5([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^5(\bar\Omega)}\big]\big(\|f_L\|_{C^2(S^1)} + \|f_R\|_{C^2(S^1)}\big). \tag{3.78}$$

Together with (3.32), we conclude that

$$\|\varphi_0\|_{C^3(\bar\Omega)} \leq C\big[\|\boldsymbol{x}\|_{C^5([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^5(\bar\Omega)}\big]\|\psi^e_0\|_{C^5(\bar\Omega)}. \tag{3.79}$$

where $\varphi_0 = \tilde\varphi_0 \circ \boldsymbol{x}^{-1}$.

Overall, we have the following refined estimate

$$\omega^a_0 = \begin{cases} \omega^e_0 + \Big(R_4(\psi^e_0) + \triangle\varphi_0\Big)\overset{\times}{h}{}^2 + \Big(R^h_5(\psi^e_0) + R^h_3(\varphi_0)\Big)\overset{\times}{h}{}^3 & \text{on } \Gamma_g \\ \omega^e_0 + \Big(R_4(\psi^e_0) + \triangle\varphi_0\Big)\overset{\times}{h}{}^2 + \Big(\mathring{R}^h_5(\psi^e_0) + \mathring{R}^h_3(\varphi_0)\Big)\overset{\times}{h}{}^3 & \text{on } \mathring{\Omega}_g, \end{cases} \tag{3.80}$$

where

$$R_4(\psi^e_0) = \frac{1}{24\sqrt{\overset{\times}{g}}}\sum_{\gamma=1}^{2}\Big(\overset{\times}{\partial}{}^3_\gamma\big(\sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{1\gamma}\overset{\times}{\partial}_1\psi^e + \sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{2\gamma}\overset{\times}{\partial}_2\psi^e\big) + \overset{\times}{\partial}_\gamma\big(\sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{1\gamma}\overset{\times}{\partial}{}^3_1\psi^e + \sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{2\gamma}\overset{\times}{\partial}{}^3_2\psi^e\big)\Big) \tag{3.81}$$

and the remainder terms satisfy

$$\|R^h_k(\cdot)\|_{C^0(\bar\Omega)},\ \|\mathring{R}^h_k(\cdot)\|_{C^0(\bar\Omega)} \leq C\big[\|\boldsymbol{x}\|_{C^k([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^k(\bar\Omega)}\big]\|\cdot\|_{C^k(\bar\Omega)}. \tag{3.82}$$

It is easy to see from (3.80)-(3.82) that

$$\|\overset{\times}{\nabla}{}^\perp_h(\omega^a_0 - \omega^e_0)\|_{\Omega_c} \leq C_{10}[\|\boldsymbol{x}\|_{C^5([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^5(\bar\Omega)}, \|\psi^e_0\|_{C^5(\bar\Omega)}]\overset{\times}{h}{}^2. \tag{3.83}$$

In view of Remark 1 and the analysis above, the estimates (3.67) and (3.68) result in full second order accuracy provided the initial data is chosen properly. In particular, we have the following

Corollary 1. *Assume the mapping* $\boldsymbol{x} : (\xi^1, \xi^2) \mapsto (x, y)$ *is a* C^5 *bijection from* $[0,1] \times S^1$ *to* $\bar{\Omega} \subset \mathbb{R}^2$ *and* $\boldsymbol{u}_0^h = \boldsymbol{u}_0^a$*. Then*

$$\max_{[0,T]} \|\partial_t(\boldsymbol{u}^h - \boldsymbol{u}^e)\|_{\Omega_c}^2 + \nu \int_0^T \|\partial_t(\omega^h - \omega^e)\|_{\bar{\Omega}_g}^2 + \max_{[0,T]} \|p^h - p^e\|_{\bar{\Omega}_g}^2 \le K_7 \overset{\times}{h}{}^4, \tag{3.84}$$

where $K_7 = K_7\big[T, \nu, \|\boldsymbol{u}^e\|_{H^1(0,T;C^4(\bar{\Omega}))}, \|\boldsymbol{u}^e\|_{H^2(0,T;C^2(\bar{\Omega}))} \|p^e\|_{H^1(0,T;C^3(\bar{\Omega}))}, \|\boldsymbol{x}\|_{C^5([0,1]\times S^1)}, \|\boldsymbol{x}^{-1}\|_{C^5(\bar{\Omega})}\big]$.

It is worth noting that the extra regularity requirement on $\boldsymbol{x}$ is only needed in constructing a good initial data. It will be interesting to see if there exists initial data satisfying (3.83) under a C^4 bijection.

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Appendix A. Proof of Theorem 2, the LBB Condition.

In this appendix, we proceed with the proof of Theorem 2, the LBB condition for GMAC.

Our proof consists of two parts. In section A.1, Theorem 5, we will prove a (formally) stronger version of the LBB condition for the special case of conformal metrics. That is, mappings with corresponding metric tensors of the form

$$\underset{+}{g}_{11} = \underset{+}{g}_{22} = \sqrt{\underset{+}{g}}(\xi^1, \xi^2), \quad \underset{+}{g}_{12} = \underset{+}{g}_{21} \equiv 0. \tag{A.1}$$

For example, the mapping $\underset{+}{\boldsymbol{x}} : (\xi^1, \xi^2) \mapsto (\underset{+}{x}, \underset{+}{y})$ with

$$\underset{+}{x} = e^{2\pi\xi^1}\cos(2\pi\xi^2), \qquad \underset{+}{y} = e^{2\pi\xi^1}\sin(2\pi\xi^2),$$

satisfies (A.1) with $\sqrt{\underset{+}{g}}(\xi^1, \xi^2) = 4\pi^2 e^{4\pi\xi^1}$.

To make distinction between conformal and general metrics, we will add the subscript '+' for all quantities derived from the conformal coordinate mapping, including the basis vectors and the difference operators and various norms in the rest of the paper.

This strong LBB condition, together with a crucial estimate (Lemma A.5), are then used to give a constructive proof for the general case in section A.2. Note that we do *not* require the general case to be small perturbation of the conformal one.

A.1. Strong Form of LBB Condition for Conformal Metrics. We start with the following strong version of the LBB condition for conformal metrics satisfying (A.1).

THEOREM 5 (**Strong form of LBB**). *If the mapping $\underset{+}{\boldsymbol{x}} : (\xi^1, \xi^2) \mapsto (\underset{+}{x}, \underset{+}{y})$ is a conformal bijection between $[0,1] \times S^1$ and $\underset{+}{\bar{\Omega}} \subset \mathbb{R}^2$, then given any $q \in L^2(\underset{+}{\bar{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$, there exists a vector field $\boldsymbol{v} \in L^2(\underset{+}{\Omega}_{\mathrm{c}}, \mathbb{R}^2)$ such that*

$$\underset{+}{\overset{\times}{\nabla}}{}'_h \cdot \boldsymbol{v} = q \tag{A.2}$$

and

$$\gamma \|\boldsymbol{v}\|_{\underset{+}{\overset{\times}{H}}{}^1_h} \le \|q\|_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}} \tag{A.3}$$

where $\gamma > 0$ is a constant independent of q, $\boldsymbol{v}$ or $\overset{\times}{h}$.

It is easy to see that (A.2, A.3) implies the standard LBB condition

$$\inf_{q \in L^2(\underset{+}{\bar{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2} \ \sup_{\boldsymbol{v} \in L^2(\underset{+}{\Omega}_{\mathrm{c}}, \mathbb{R}^2)} \frac{\langle\, q \,,\, \underset{+}{\overset{\times}{\nabla}}{}'_h \cdot \boldsymbol{v} \,\rangle_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}}}{\|q\|_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}} \|\boldsymbol{v}\|_{\underset{+}{\overset{\times}{H}}{}^1_h}} \ge \gamma \quad \text{uniformly in } h. \tag{A.4}$$

In fact, it can be shown that the strong form of LBB condition (A.2, A.3) is equivalent to the standard LBB condition (A.4).

Our approach for Theorem 5 is based on a global construction procedure using Fourier series. We start with a list of notations. Define

$$\mathbf{C}_i^m \triangleq \begin{cases} \sqrt{2}\cos(m\pi\xi_i^1), & 1 \le m \le N-1; \\ \cos(m\pi\xi_i^1), & m = 0, N, \end{cases} \tag{A.5}$$

$$\mathbf{S}_i^m \triangleq \sqrt{2}\sin(m\pi\xi_i^1), \qquad 1 \le m \le N-1, \tag{A.6}$$

$$\mathbf{E}_j^n \triangleq \exp(2n\pi\sqrt{-1}\xi_j^2), \qquad 0 \le n \le N-1. \tag{A.7}$$

It is easy to verify that

$$\{\mathbf{C}^m \otimes \mathbf{E}^n \,|\, 0 \le m \le N,\ 0 \le n \le N-1\} \tag{A.8}$$

is an orthonormal basis for $L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})$ with respect to the standard inner product

$$\langle a, b\rangle_{0,\bar{\underset{+}{\Omega}}_{\mathrm{g}}} = h^2 \sum_{i=0}^{N}{}' \sum_{j=1}^{N} (ab)_{ij}. \tag{A.9}$$

where the primed sum denotes half weight at $i = 0$ and $i = N$. In addition, (A.8) is an orthonormal eigen-basis for $\sqrt{\underset{+}{g}}\overset{\times}{\underset{+}{\triangle}}{}'_h$. Indeed, since the coordinate mapping is conformal and $h_1 = h_2 = h$, it follows that $\overset{\times}{\underset{+}{g}}{}^{11} = \overset{\times}{\underset{+}{g}}{}^{22} = 1/\sqrt{\overset{\times}{\underset{+}{g}}} = 1/\sqrt{\underset{+}{g}}$ and $\overset{\times}{\underset{+}{g}}{}^{12} = \overset{\times}{\underset{+}{g}}{}^{21} \equiv 0$. Thus

$$\overset{\times}{\underset{+}{\triangle}}{}'_h = \frac{1}{\sqrt{\underset{+}{g}}}\big(\overset{\times}{D}{}_1^{2\prime} + \overset{\times}{D}{}_2^{2\prime}\big) = \frac{1}{\sqrt{\underset{+}{g}}}\big(D_1^{2\prime} + D_2^2 + \frac{h^2}{2} D_1^{2\prime} D_2^2\big). \tag{A.10}$$

where

$$(D_1^{2\prime} f)_i = \begin{cases} \frac{2}{h^2}(f_1 - f_0) & i = 0 \\ \frac{1}{h^2}(f_{i+1} - 2f_i + f_{i-1}) & 1 \le i \le N-1 \\ \frac{2}{h^2}(f_{N-1} - f_N) & i = N \end{cases}, \quad (D_2^2 g)_j = \frac{g_{j+1} - 2g_j + g_{j-1}}{h^2}, \quad 1 \le j \le N. \tag{A.11}$$

The reduction is not needed in the ξ^2 direction due to periodicity.

It is east to see that

$$\begin{aligned} D_1^{2\prime}\mathbf{C}_i^m &= -\lambda_m^2 \mathbf{C}_i^m, \quad 0 \le i \le N, \\ D_2^2 \mathbf{E}_j^n &= -\lambda_{2n}^2 \mathbf{E}_j^n, \quad\ 1 \le j \le N, \end{aligned} \tag{A.12}$$

where

$$\lambda_m \triangleq \frac{2\sin(m\pi h/2)}{h}. \tag{A.13}$$

Therefore

$$\sqrt{\underset{+}{g}}\overset{\times}{\underset{+}{\triangle}}{}'_h (\mathbf{C}^m \otimes \mathbf{E}^n)_{ij} = -\overset{\times}{\kappa}{}^2_{mn} (\mathbf{C}^m \otimes \mathbf{E}^n)_{ij}, \tag{A.14}$$

where

$$\overset{\times}{\kappa}{}^2_{mn} \triangleq \lambda_m^2 + \lambda_{2n}^2 - \frac{h^2}{2}\lambda_m^2 \lambda_{2n}^2 = \frac{4}{h^2}\big(\sin^2(\frac{m\pi h}{2})\cos^2(n\pi h) + \cos^2(\frac{m\pi h}{2})\sin^2(n\pi h)\big). \tag{A.15}$$

On the other hand,

$$\{\mathbf{S}^m \otimes \mathbf{E}^n \,|\, 1 \le m \le N-1,\ 0 \le n \le N-1\} \tag{A.16}$$

is an orthonormal basis for

$$L_0^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R}) \triangleq \big\{\psi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R}) \,\big|\, \psi_{0,j} = 0 = \psi_{N,j},\ 1 \le j \le N\big\} \tag{A.17}$$

with

$$D_1^2 \mathbf{S}_i^m = -\lambda_m^2 \mathbf{S}_i^m, \qquad 1 \le i \le N-1, \tag{A.18}$$

and

$$\sqrt{\underset{+}{g}}\,\overset{\times}{\underset{+}{\triangle}}_h (\mathbf{S}^m \otimes \mathbf{E}^n)_{ij} = -\overset{\times}{\kappa}{}^2_{mn} (\mathbf{S}^m \otimes \mathbf{E}^n)_{ij}, \qquad 1 \le i \le N-1. \tag{A.19}$$

Note that, however, (A.16) is not an eigen-basis for $\sqrt{\underset{+}{g}}\,\overset{\times}{\underset{+}{\triangle}}{}'_h$ since $\sqrt{\underset{+}{g}}\,\overset{\times}{\underset{+}{\triangle}}{}'_h (\mathbf{S}^m \otimes \mathbf{E}^n) \neq 0$ on Γ_{g}.

Proof of Theorem 5. Given $q \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$, we will construct explicitly a vector field $\boldsymbol{v}$ that satisfies (A.2) and (A.3). This is done in the following steps:

Step 1:

Solve for $\phi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$ from

$$\overset{\times}{\underset{+}{\triangle}}{}'_h \phi = q \qquad \text{on } \bar{\underset{+}{\Omega}}_{\mathrm{g}}. \tag{A.20}$$

From (A.10-A.15) above, it is easy to see that we can solve (A.20) by expanding ϕ and $Q = \sqrt{\underset{+}{g}}q$ with respect to the eigen-basis (A.8),

$$\phi_{ij} = \sum_{n=0}^{N-1} \sum_{m=0}^{N} \hat{\phi}_{mn} \mathbf{C}_i^m \mathbf{E}_j^n, \qquad Q_{ij} = (\sqrt{\underset{+}{g}}q)_{ij} = \sum_{n=0}^{N-1} \sum_{m=0}^{N} \hat{Q}_{mn} \mathbf{C}_i^m \mathbf{E}_j^n. \tag{A.21}$$

and compare the coefficients mode by mode to get

$$\hat{\phi}_{mn} = \begin{cases} -\dfrac{1}{\overset{\times}{\kappa}{}^2_{mn}} \hat{Q}_{mn}, & (m,n) \neq (0,0), (N, \frac{N}{2}) \\ 0, & \text{otherwise.} \end{cases} \tag{A.22}$$

Note that from (A.15), $\overset{\times}{\kappa}{}^2_{mn} = 0$ if and only if $(m,n) = (0,0)$ or $(N, \frac{N}{2})$, while $\hat{Q}_{0,0} = \hat{Q}_{N,\frac{N}{2}} = 0$ as $q \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$. Thus (A.22) indeed gives the unique solution ϕ in $L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$. It is worth mentioning that, from the analysis above, we have for $\phi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$,

$$\begin{aligned} h^2 \sum_{i=0}^{N}{}' \sum_{j=1}^{N} \phi_{ij}^2 &= \sum_{m=0}^{N} \sum_{n=0}^{N-1} |\hat{\phi}_{mn}|^2 \le \frac{1}{\overset{\times}{\kappa}{}^4_{\min}} \sum_{m=0}^{N} \sum_{n=0}^{N-1} |\overset{\times}{\kappa}{}^2_{mn} \hat{\phi}_{mn}|^2 \\ &= \frac{h^2}{\overset{\times}{\kappa}{}^4_{\min}} \sum_{i=0}^{N}{}' \sum_{j=1}^{N} (\sqrt{\underset{+}{g}}\,\overset{\times}{\underset{+}{\triangle}}{}'_h \phi)^2_{ij} \end{aligned} \tag{A.23}$$

where

$$\overset{\times}{\kappa}{}^2_{\min} \triangleq \min_{(m,n)\neq(0,0),(N,\frac{N}{2})} \overset{\times}{\kappa}{}^2_{mn} \tag{A.24}$$

Moreover, with straightforward calculation (see also (A.76, A.80) below), it is easy to see that

$$\overset{\times}{\kappa}{}^2_{\min} = \overset{\times}{\kappa}{}^2_{1,0} = \frac{4}{h^2} \sin^2\left(\frac{\pi h}{2}\right) = O(1), \qquad 8 \le \overset{\times}{\kappa}{}^2_{\min} < \pi^2. \tag{A.25}$$

Thus we have the following estimate for the solution $\phi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$:

$$\|\phi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}} \leq \frac{\overset{\times}{\underset{+}{g}}_{\max}}{\overset{\times}{\kappa}^4_{\min}} \|\overset{\times}{\underset{+}{\triangle}}{}'_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}}. \tag{A.26}$$

Following similar calculations, one can also show that

$$h^2 {\sum_{i=0}^{N}}' \sum_{j=1}^{N} \phi^2_{ij} = \sum_{m=0}^{N} \sum_{n=0}^{N-1} |\hat{\phi}_{mn}|^2 \leq \frac{1}{\overset{\times}{\kappa}^2_{\min}} \sum_{m=0}^{N} \sum_{n=0}^{N-1} \overset{\times}{\kappa}^2_{mn} |\hat{\phi}_{mn}|^2 = \frac{1}{\overset{\times}{\kappa}^2_{\min}} \|\overset{\times}{\underset{+}{\nabla}}_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} \tag{A.27}$$

and

$$\|\overset{\times}{\underset{+}{\nabla}}_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} = \sum_{m=0}^{N} \sum_{n=0}^{N-1} \overset{\times}{\kappa}^2_{mn} |\hat{\phi}_{mn}|^2 \leq \frac{1}{\overset{\times}{\kappa}^2_{\min}} \sum_{m=0}^{N} \sum_{n=0}^{N-1} |\overset{\times}{\kappa}^2_{mn} \hat{\phi}_{mn}|^2 = \frac{h^2}{\overset{\times}{\kappa}^2_{\min}} {\sum_{i=0}^{N}}' \sum_{j=1}^{N} (\sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\triangle}}{}'_h \phi)^2_{ij}. \tag{A.28}$$

Similarly, for $\psi \in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})$, we have

$$h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N} \psi^2_{ij} = \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} |\hat{\psi}_{mn}|^2 \leq \frac{1}{\overset{\times}{\kappa}^2_{\min}} \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} \overset{\times}{\kappa}^2_{mn} |\hat{\psi}_{mn}|^2 = \frac{1}{\overset{\times}{\kappa}^2_{\min}} \|\sqrt{\underset{+}{g}}^{-\frac{1}{2}} \overset{\times}{\underset{+}{\nabla}}{}^{\perp}_h \psi\|^2_{\Omega_{\mathrm{c}}} \tag{A.29}$$

$$\|\sqrt{\underset{+}{g}}^{-\frac{1}{2}} \overset{\times}{\underset{+}{\nabla}}{}^{\perp}_h \psi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} = \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} \overset{\times}{\kappa}^2_{mn} |\hat{\psi}_{mn}|^2 \leq \frac{1}{\overset{\times}{\kappa}^2_{\min}} \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} |\overset{\times}{\kappa}^2_{mn} \hat{\psi}_{mn}|^2 = \frac{h^2}{\overset{\times}{\kappa}^2_{\min}} \sum_{i=1}^{N-1} \sum_{j=1}^{N} (\sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\triangle}}_h \psi)^2_{ij}. \tag{A.30}$$

Here in (A.29, A.30), $\hat{\psi}_{mn}$ is the coefficient with respect to the basis (A.16):

$$\psi_{ij} = \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} \hat{\psi}_{mn} \mathbf{S}^m_i \mathbf{E}^n_j, \qquad \psi \in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R}). \tag{A.31}$$

As a result, we have the following Poincaré type inequalities for $\phi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$ and $\psi \in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})$ respectively:

Lemma A.1.

1. *Let $\phi \in L^2(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$, then*

$$\|\phi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}} \leq \frac{\sqrt{\underset{+}{g}}_{\max}}{\overset{\times}{\kappa}^2_{\min}} \|\overset{\times}{\underset{+}{\nabla}}_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}}, \qquad \|\overset{\times}{\underset{+}{\nabla}}_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} \leq \frac{\sqrt{\underset{+}{g}}_{\max}}{\overset{\times}{\kappa}^2_{\min}} \|\overset{\times}{\underset{+}{\triangle}}{}'_h \phi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}}. \tag{A.32}$$

2. *Let $\psi \in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})$, then*

$$\|\psi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}} \leq \frac{\sqrt{\underset{+}{g}}_{\max}}{\overset{\times}{\kappa}^2_{\min} \sqrt{\underset{+}{g}}_{\min}} \|\overset{\times}{\underset{+}{\nabla}}{}^{\perp}_h \psi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}}, \qquad \|\overset{\times}{\underset{+}{\nabla}}{}^{\perp}_h \psi\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} \leq \frac{\overset{\times}{\underset{+}{g}}_{\max}}{\overset{\times}{\kappa}^2_{\min}} \|\overset{\times}{\underset{+}{\triangle}}_h \psi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}}. \tag{A.33}$$

Note that in (A.33),

$$\|\overset{\times}{\underset{+}{\triangle}}_h \psi\|^2_{\underset{+}{\Omega}_{\mathrm{g}}} \triangleq h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N} \left(\sqrt{\underset{+}{g}}\, (\overset{\times}{\underset{+}{\triangle}}_h \psi)^2 \right)_{ij}, \tag{A.34}$$

the integrand is summed over interior grids only. In general, $\overset{\times}{\underset{+}{\triangle}}{}_h'\psi|_{\Gamma_{\rm g}} \neq 0$ even if $\psi \in L_0^2(\underset{+}{\bar\Omega}_{\rm g}, \mathbb{R})$.

Step 2:

With ϕ obtained in step 1, construct $\psi \in L_0^2(\underset{+}{\bar\Omega}_{\rm g}, \mathbb{R})$ such that

$$D_n^- \psi\big|_{\Gamma_{\rm g}} = \tilde D_\tau \phi\big|_{\mathring{\Gamma}_{\rm g}} \tag{A.35}$$

and

$$\|\overset{\times}{\underset{+}{\triangle}}{}_h \psi\|_{\underset{+}{\mathring\Omega}_{\rm g}} \le C \|\overset{\times}{\underset{+}{\triangle}}{}_h' \phi\|_{\underset{+}{\bar\Omega}_{\rm g}}, \tag{A.36}$$

where $\mathring{\Gamma}_{\rm g} = \{(\xi_i^1, \xi_j^2)\,|\, i = 1 \text{ or } N-1,\, 1 \le j \le N\}$, C is a constant that only depends on $\sqrt{\underset{+}{g}}_{\max}$ and $\sqrt{\underset{+}{g}}_{\min}$, D_n^- is the one-sided backward difference with respect to the unit outer normal $\boldsymbol{n}$,

$$D_n^- \psi_{i,j} = \begin{cases} \dfrac{\psi_{0,j} - \psi_{1,j}}{h}, & i = 0, \\[2mm] \dfrac{\psi_{N,j} - \psi_{N-1,j}}{h}, & i = N; \end{cases} \tag{A.37}$$

and $\tilde D_\tau$ the long-stencil centered difference with respect to the counter-clockwise unit tangent $\boldsymbol{\tau}$,

$$\tilde D_\tau \phi_{i,j} = \begin{cases} \dfrac{\phi_{1,j-1} - \phi_{1,j+1}}{2h}, & i = 1, \\[2mm] \dfrac{\phi_{N-1,j+1} - \phi_{N-1,j-1}}{2h}, & i = N-1. \end{cases} \tag{A.38}$$

This step is the most technical part of the Theorem. We will detail it in section A.1.1.

Step 3:

With ψ given by Step 2, construct explicitly the vector field $\boldsymbol{v}$ as

$$\boldsymbol{v} \triangleq \overset{\times}{\underset{+}{\nabla}}_h \phi - \overset{\times}{\underset{+}{\nabla}}{}_h^\perp \psi. \tag{A.39}$$

Since $\psi \in L_0^2(\underset{+}{\bar\Omega}_{\rm g}, \mathbb{R})$, it follows from (2.33) that $\overset{\times}{\underset{+}{\nabla}}{}_h' \cdot \overset{\times}{\underset{+}{\nabla}}{}_h^\perp \psi \equiv 0$ on $\underset{+}{\bar\Omega}_{\rm g}$. Therefore from (2.31), we have

$$\overset{\times}{\underset{+}{\nabla}}{}_h' \cdot \boldsymbol{v} = \overset{\times}{\underset{+}{\nabla}}{}_h' \cdot \overset{\times}{\underset{+}{\nabla}}_h \phi - \overset{\times}{\underset{+}{\nabla}}{}_h' \cdot \overset{\times}{\underset{+}{\nabla}}{}_h^\perp \psi = \overset{\times}{\underset{+}{\triangle}}{}_h' \phi = q \qquad \text{on } \underset{+}{\bar\Omega}_{\rm g}. \tag{A.40}$$

This gives (A.2).

To see that $\boldsymbol{v}$ indeed satisfies (A.3), we first note from (A.32), (A.33), (A.36) and (A.20) that

$$\|\boldsymbol{v}\|^2_{\underset{+}{\Omega}_{\rm c}} = \|\overset{\times}{\underset{+}{\nabla}}_h \phi\|^2_{\underset{+}{\Omega}_{\rm c}} + \|\overset{\times}{\underset{+}{\nabla}}{}_h^\perp \psi\|^2_{\underset{+}{\Omega}_{\rm c}} \le \frac{1}{\overset{\times}{\kappa}{}^2_{\min}} \big(\sqrt{\underset{+}{g}}_{\max} \|\overset{\times}{\underset{+}{\triangle}}{}_h' \phi\|^2_{\underset{+}{\bar\Omega}_{\rm g}} + \underset{+}{g}_{\max} \|\overset{\times}{\underset{+}{\triangle}}{}_h \psi\|^2_{\underset{+}{\mathring\Omega}_{\rm g}}\big) \le \beta_1 \|q\|^2_{\underset{+}{\bar\Omega}_{\rm g}} \tag{A.41}$$

for some constant $\beta_1 > 0$ that only depends on $\sqrt{\underset{+}{g}}_{\max}$. On the other hand, from (2.31) and (2.32), it is easy to see that

$$\sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\nabla}}{}_h^{\perp\prime} \cdot \boldsymbol{v} = \sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\nabla}}{}_h^{\perp\prime} \cdot (\overset{\times}{\underset{+}{\nabla}}_h \phi - \overset{\times}{\underset{+}{\nabla}}{}_h^\perp \psi) = \sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\triangle}}{}_h' \psi \qquad \text{on } \underset{+}{\mathring\Omega}_{\rm g}. \tag{A.42}$$

Note that in general, $\sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\nabla}}{}_h^{\perp\prime} \cdot \overset{\times}{\underset{+}{\nabla}}_h \phi \neq 0$ on $\Gamma_{\rm g}$. The net contribution of $\sqrt{\underset{+}{g}}\, \overset{\times}{\underset{+}{\nabla}}{}_h^{\perp\prime} \cdot \boldsymbol{v}$ on (ξ_0^1, ξ_j^2) for example, can be calculated by applying the boundary condition (A.35), that is,

$$\psi_{1,j} = \frac{\phi_{1,j+1} - \phi_{1,j-1}}{2}, \tag{A.43}$$

to get

$$
\begin{aligned}
(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \boldsymbol{v})_{0,j} =&(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \underset{+}{\overset{\times}{\nabla}}{}_h\phi - \sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \underset{+}{\overset{\times}{\nabla}}{}_h^{\perp}\psi)_{0,j}\\
=&2\,\frac{\phi_{0,j+1}-\phi_{0,j-1}-\psi_{1,j+1}-\psi_{1,j-1}}{\overset{\times}{h}{}^2}\\
=&\frac{2\phi_{0,j+1}-2\phi_{0,j-1}-(\phi_{1,j+2}-\phi_{1,j})-(\phi_{1,j}-\phi_{1,j-2})}{\overset{\times}{h}{}^2}\\
=&\frac{-(\phi_{1,j+2}-2\phi_{0,j+1}+\phi_{1,j})+(\phi_{1,j}-2\phi_{0,j-1}+\phi_{1,j-2})}{\overset{\times}{h}{}^2}\\
=&\frac{(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\triangle}}{}_h^{\prime}\phi)_{0,j-1}-(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\triangle}}{}_h^{\prime}\phi)_{0,j+1}}{2} = \frac{1}{2}\Big((\sqrt{\underset{+}{g}}q)_{0,j-1}-(\sqrt{\underset{+}{g}}q)_{0,j+1}\Big).
\end{aligned}
\tag{A.44}
$$

The calculation of $(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \boldsymbol{v})_{N,j}$ is similar.

In summary, we have

$$
(\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \boldsymbol{v})_{i,j} = \begin{cases} (\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\triangle}}{}_h\psi)_{i,j} & 1\le i\le N-1;\\ \frac{1}{2}\big((\sqrt{\underset{+}{g}}q)_{0,j-1}-(\sqrt{\underset{+}{g}}q)_{0,j+1}\big) & i=0;\\ \frac{1}{2}\big((\sqrt{\underset{+}{g}}q)_{N,j+1}-(\sqrt{\underset{+}{g}}q)_{N,j-1}\big) & i=N.\end{cases}
\tag{A.45}
$$

It follows from (A.45), (A.36) and (A.20) that

$$
\|\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot \boldsymbol{v}\|^2_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}} \le \beta_2\|q\|^2_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}}
\tag{A.46}
$$

where β_2 is a constant that only depends on $\sqrt{\underset{+}{g}}$. In view of (A.40), (A.41) and (A.46), the estimate (A.3) follows with $\gamma^{-1}=\sqrt{\beta_1+\beta_2+1}$. This completes the proof of Theorem 5. ∎

A.1.1. Construction and Estimate of ψ. We now proceed with detailed construction and estimate for the potential ψ asserted in Step 2.

We first define the $\overset{\times}{H}{}_h^{\frac{1}{2}}$ norm for periodic grid functions which is essential to our analysis. Let f be a periodic grid function on $\{\xi_0,\xi_1,\cdots,\xi_N\equiv\xi_0\}$, we can expand $f(\xi_j)=\sum_{n=0}^{N-1}\hat{f}_n\,\mathbf{E}_j^n$ and define

$$
\|f\|^2_{\overset{\times}{H}{}_h^{\frac{1}{2}}} \triangleq \sum_{n=0}^{N-1}|\hat{f}_n|^2\overset{\times}{\mu}_n
\tag{A.47}
$$

and for $\phi\in L^2(\underset{+}{\bar{\Omega}}_{\mathrm{g}},\mathbb{R})/\mathbb{R}^2$,

$$
\|\tilde{D}_\tau\phi\|^2_{\overset{\times}{H}{}_h^{\frac{1}{2}}(\mathring{\Gamma}_{\mathrm{g}})} \triangleq \|\tilde{D}_2\phi(\xi_1^1,\cdot)\|^2_{\overset{\times}{H}{}_h^{\frac{1}{2}}} + \|\tilde{D}_2\phi(\xi_{N-1}^1,\cdot)\|^2_{\overset{\times}{H}{}_h^{\frac{1}{2}}},
\tag{A.48}
$$

where

$$
\overset{\times}{\mu}_n \triangleq \Big(\sum_{m\in\overset{\times}{\mathbb{M}}_n}\frac{\cos^2(m\pi h)}{\overset{\times}{\kappa}{}^2_{mn}}\Big)^{-1}
\tag{A.49}
$$

and

$$
\overset{\times}{\mathbb{M}}_n \triangleq \begin{cases}\{\,1,2,\cdots,N\,\} & n=0;\\ \{\,0,1,\cdots,N-1\,\} & n=N/2;\\ \{\,0,1,\cdots,N\,\} & \text{otherwise.}\end{cases}
\tag{A.50}
$$

is where $\overset{\times}{\kappa}_{mn} \neq 0$.

We have the following discrete analogue of the trace inequality:

LEMMA A.2. *Let $\phi \in L^2(\bar{\Omega}_{+\mathrm{g}}, \mathbb{R})/\mathbb{R}^2$. Then*

$$\|\tilde{D}_\tau \phi\|_{\overset{\times}{H}_h^{\frac{1}{2}}(\mathring{\Gamma}_{\mathrm{g}})} \leq 4\sqrt{g}_{+\max} \|\overset{\times}{\triangle}{}'_{+h} \phi\|_{\bar{\Omega}_{+\mathrm{g}}}. \tag{A.51}$$

Proof. We first show that

$$\|\tilde{D}_2 \phi(\xi_1^1, \cdot)\|^2_{\overset{\times}{H}_h^{\frac{1}{2}}} \leq 2\sqrt{g}_{+\max} \|\overset{\times}{\triangle}{}'_{+h} \phi\|^2_{\bar{\Omega}_{+\mathrm{g}}}. \tag{A.52}$$

Since

$$\frac{\mathbf{E}_{j+1}^n - \mathbf{E}_{j-1}^n}{2h} = \tilde{\lambda}_{2n} \mathbf{E}_j^n, \qquad \tilde{\lambda}_{2n} \triangleq \frac{\sin(2n\pi h)}{h}. \tag{A.53}$$

It follows that

$$\tilde{D}_2 \phi(\xi_1^1, \xi_j^2) = \sum_{n=0}^{N-1} \Big(\sum_{m=0}^{N} \sqrt{-1} \tilde{\lambda}_{2n} \hat{\phi}_{mn} \mathbf{C}_1^m \Big) \mathbf{E}_j^n. \tag{A.54}$$

Hence

$$\begin{aligned}
\|\tilde{D}_2 \phi(\xi_1^1, \cdot)\|^2_{\overset{\times}{H}_h^{\frac{1}{2}}} &= \sum_{n=0}^{N-1} \Big| \sum_{m=0}^{N} \tilde{\lambda}_{2n} \hat{\phi}_{mn} \mathbf{C}_1^m \Big|^2 \overset{\times}{\mu}_n \\
&= \sum_{n=0}^{N-1} \Big| \sum_{m \in \overset{\times}{\mathbb{M}}} \frac{\tilde{\lambda}_{2n} \hat{\phi}_{mn} \mathbf{C}_1^m \overset{\times}{\kappa}_{mn}}{\overset{\times}{\kappa}_{mn}} \Big|^2 \overset{\times}{\mu}_n \\
&\leq 2 \sum_{n=0}^{N-1} \Big(\sum_{m=0}^{N} \overset{\times}{\kappa}{}^2_{mn} \tilde{\lambda}^2_{2n} |\hat{\phi}_{mn}|^2 \Big) \Big(\sum_{m \in \overset{\times}{\mathbb{M}}} \frac{\cos^2(m\pi h)}{\overset{\times}{\kappa}{}^2_{mn}} \Big) \overset{\times}{\mu}_n \\
&= 2 \sum_{n=0}^{N-1} \sum_{m=0}^{N} \overset{\times}{\kappa}{}^2_{mn} \tilde{\lambda}^2_{2n} |\hat{\phi}_{mn}|^2
\end{aligned} \tag{A.55}$$

where we have used (A.49) in the last equality.

Since

$$\begin{aligned}
\tilde{\lambda}^2_{2n} &= \frac{4}{h^2} \sin^2(n\pi h) \cos^2(n\pi h) \\
&\leq \frac{4}{h^2} \min\{\sin^2(n\pi h), \cos^2(n\pi h)\} \\
&\leq \frac{4}{h^2} \Big(\cos^2\big(\frac{m\pi h}{2}\big) \sin^2(n\pi h) + \sin^2\big(\frac{m\pi h}{2}\big) \cos^2(n\pi h) \Big) = \overset{\times}{\kappa}{}^2_{mn},
\end{aligned} \tag{A.56}$$

it follows from (A.55) and (A.56) that

$$\|\tilde{D}_2 \phi(\xi_1^1, \cdot)\|^2_{\overset{\times}{H}_h^{\frac{1}{2}}} \leq 2 \sum_{n=0}^{N-1} \sum_{m=0}^{N} \overset{\times}{\kappa}{}^4_{mn} |\hat{\phi}_{mn}|^2 \leq 2\sqrt{g}_{+\max} \|\overset{\times}{\triangle}{}'_{+h} \phi\|^2_{\bar{\Omega}_{+\mathrm{g}}} \tag{A.57}$$

The estimate for $\|\tilde{D}_2\phi(\xi^1_{N-1},\cdot)\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h}$ is similar and the proof is complete. ∎

We proceed with the construction of the potential $\psi(\xi^1,\xi^2)$. Let

$$
\begin{aligned}
a^{\mathrm{e}}(\xi^2_j) &\triangleq \frac{\tilde{D}_2\phi_{1,j}+\tilde{D}_2\phi_{N-1,j}}{2} = \sum_{n=0}^{N-1}\hat{a}^{\mathrm{e}}_n\mathbf{E}^n_j,\\
a^{\mathrm{o}}(\xi^2_j) &\triangleq \frac{\tilde{D}_2\phi_{1,j}-\tilde{D}_2\phi_{N-1,j}}{2} = \sum_{n=0}^{N-1}\hat{a}^{\mathrm{o}}_n\mathbf{E}^n_j,
\end{aligned}
\tag{A.58}
$$

and define

$$
\overset{\times}{\sigma}{}^{\mathrm{e}}_n \triangleq \Big(\sum_{\substack{m=1\\ m\text{ is even}}}^{N-1}\frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}}\Big)^{-1},\qquad \overset{\times}{\sigma}{}^{\mathrm{o}}_n \triangleq \Big(\sum_{\substack{m=1\\ m\text{ is odd}}}^{N-1}\frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}}\Big)^{-1}
\tag{A.59}
$$

The proposed $\psi\in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}},\mathbb{R})$ is given by

$$
\psi(\xi^1_i,\xi^2_j)=\sum_{m=1}^{N-1}\sum_{n=0}^{N-1}\hat{\psi}_{mn}\mathbf{S}^m_i\mathbf{E}^n_j
\tag{A.60}
$$

where

$$
\hat{\psi}_{mn}\triangleq\begin{cases}\hat{a}^{\mathrm{e}}_n\dfrac{\tilde{\lambda}_m\overset{\times}{\sigma}{}^{\mathrm{e}}_n}{\sqrt{2}\overset{\times}{\kappa}{}^4_{mn}}, & \text{if } m \text{ is even;}\\[2ex] \hat{a}^{\mathrm{o}}_n\dfrac{\tilde{\lambda}_m\overset{\times}{\sigma}{}^{\mathrm{o}}_n}{\sqrt{2}\overset{\times}{\kappa}{}^4_{mn}}, & \text{if } m \text{ is odd.}\end{cases}
\tag{A.61}
$$

It is easy to verify that the constructed ψ satisfies (A.35):

$$
\begin{aligned}
-D^-_n\psi(0,\xi^2_j) =& D^+_1\psi(0,\xi^2_j)=\frac{\psi(\xi^1_1,\xi^2_j)-0}{h}\\
=&\sum_{n=0}^{N-1}\Big(\sum_{\substack{m=1\\ m\text{ is even}}}^{N-1}\hat{a}^{\mathrm{e}}_n\frac{\tilde{\lambda}^2_m\overset{\times}{\sigma}{}^{\mathrm{e}}_n}{\overset{\times}{\kappa}{}^4_{mn}}+\sum_{\substack{m=1\\ m\text{ is odd}}}^{N-1}\hat{a}^{\mathrm{o}}_n\frac{\tilde{\lambda}^2_m\overset{\times}{\sigma}{}^{\mathrm{o}}_n}{\overset{\times}{\kappa}{}^4_{mn}}\Big)\mathbf{E}^n_j\\
=&\sum_{n=0}^{N-1}\Big(\hat{a}^{\mathrm{e}}_n\overset{\times}{\sigma}{}^{\mathrm{e}}_n\big(\sum_{\substack{m=1\\ m\text{ is even}}}^{N-1}\frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}}\big)+\hat{a}^{\mathrm{o}}_n\overset{\times}{\sigma}{}^{\mathrm{o}}_n\big(\sum_{\substack{m=1\\ m\text{ is odd}}}^{N-1}\frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}}\big)\Big)\mathbf{E}^n_j\\
=&\sum_{n=0}^{N-1}(\hat{a}^{\mathrm{e}}_n+\hat{a}^{\mathrm{o}}_n)\mathbf{E}^n_j=\tilde{D}_2\phi_{1,j}=-\tilde{D}_\tau\phi_{1,j}.
\end{aligned}
\tag{A.62}
$$

where we have used $\mathbf{S}^m_1=\sqrt{2}h\tilde{\lambda}_m$ in the second equality above. Similarly,

$$
D^-_n\psi(1,\xi^2_j)=D^-_1\psi(1,\xi^2_j)=\frac{0-\psi(\xi^1_{N-1},\xi^2_j)}{h}=\sum_{n=0}^{N-1}(\hat{a}^{\mathrm{e}}_n-\hat{a}^{\mathrm{o}}_n)\mathbf{E}^n_j=\tilde{D}_2\phi_{N-1,j}=\tilde{D}_\tau\phi_{N-1,j}.
\tag{A.63}
$$

In addition, the constructed potential ψ decays at designed rate in Fourier modes. This enables us to give an inverse trace estimate:

LEMMA A.3. *Let* $\psi \in L^2_0(\bar{\underset{+}{\Omega}}_{\mathrm{g}}, \mathbb{R})$ *be given by (A.60, A.61) and*

$$\|D_n^- \psi\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h(\Gamma_{\mathrm{g}})} \triangleq \|D_1^+ \psi(0,\cdot)\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h} + \|D_1^- \psi(1,\cdot)\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h}. \tag{A.64}$$

Then there is a constant C *that only depends on* $\sqrt{\underset{+}{g}}$ *such that*

$$\|\underset{+}{\overset{\times}{\triangle}}{}_h \psi\|_{\underset{+}{\mathring{\Omega}}_{\mathrm{g}}} \le C \|D_n^- \psi\|_{\overset{\times}{H}{}^{\frac{1}{2}}_h(\Gamma_{\mathrm{g}})}. \tag{A.65}$$

Proof. We first expand $\sqrt{\underset{+}{g}}\underset{+}{\overset{\times}{\triangle}}{}_h \psi$ with respect to the basis (A.16) on $\underset{+}{\mathring{\Omega}}_{\mathrm{g}}$:

$$\sqrt{\underset{+}{g}}\underset{+}{\overset{\times}{\triangle}}{}_h \psi_{i,j} = \sum_{n=0}^{N-1}\sum_{m=1}^{N-1} -\overset{\times}{\kappa}{}^2_{mn} \hat{\psi}_{mn} \mathbf{S}_i^m \mathbf{E}_j^n, \qquad 1 \le i \le N-1,\ 1 \le j \le N. \tag{A.66}$$

From (A.61), we have

$$\sqrt{\underset{+}{g}}\underset{+}{\overset{\times}{\triangle}}{}_h \psi_{i,j} = -\sum_{n=0}^{N-1}\Big(\sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \overset{\times}{\kappa}{}^2_{mn} \hat{a}^{\mathrm{e}}_n \frac{\tilde{\lambda}_m \overset{\times}{\sigma}{}^{\mathrm{e}}_n}{\sqrt{2}\overset{\times}{\kappa}{}^4_{mn}} \mathbf{S}_i^m + \sum_{\substack{m=1\\ m \text{ is odd}}}^{N-1} \overset{\times}{\kappa}{}^2_{mn} \hat{a}^{\mathrm{o}}_n \frac{\tilde{\lambda}_m \overset{\times}{\sigma}{}^{\mathrm{o}}_n}{\sqrt{2}\overset{\times}{\kappa}{}^4_{mn}} \mathbf{S}_i^m \Big)\mathbf{E}_j^n, \tag{A.67}$$

In view of (A.59), we therefore have

$$\begin{aligned}
\|\underset{+}{\overset{\times}{\triangle}}{}_h \psi\|^2_{\underset{+}{\mathring{\Omega}}_{\mathrm{g}}} &\le \frac{1}{2\sqrt{\underset{+}{g}}_{\min}} \sum_{n=0}^{N-1}\Big(\sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} (|\hat{a}^{\mathrm{e}}_n| \frac{\tilde{\lambda}_m \overset{\times}{\sigma}{}^{\mathrm{e}}_n}{\overset{\times}{\kappa}{}^2_{mn}})^2 + \sum_{\substack{m=1\\ m \text{ is odd}}}^{N-1} (|\hat{a}^{\mathrm{o}}_n| \frac{\tilde{\lambda}_m \overset{\times}{\sigma}{}^{\mathrm{o}}_n}{\overset{\times}{\kappa}{}^2_{mn}})^2\Big) \\
&= \frac{1}{2\sqrt{\underset{+}{g}}_{\min}} \sum_{n=0}^{N-1}\Big(|\hat{a}^{\mathrm{e}}_n \overset{\times}{\sigma}{}^{\mathrm{e}}_n|^2 (\sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}}) + |\hat{a}^{\mathrm{o}}_n \overset{\times}{\sigma}{}^{\mathrm{o}}_n|^2 (\sum_{\substack{m=1\\ m \text{ is odd}}}^{N-1} \frac{\tilde{\lambda}^2_m}{\overset{\times}{\kappa}{}^4_{mn}})\Big) \\
&= \frac{1}{2\sqrt{\underset{+}{g}}_{\min}} \sum_{n=0}^{N-1}\Big(|\hat{a}^{\mathrm{e}}_n|^2 \overset{\times}{\sigma}{}^{\mathrm{e}}_n + |\hat{a}^{\mathrm{o}}_n|^2 \overset{\times}{\sigma}{}^{\mathrm{o}}_n\Big) \\
&\le \frac{C_*}{2\sqrt{\underset{+}{g}}_{\min}} \sum_{n=0}^{N-1} (|\hat{a}^{\mathrm{e}}_n|^2 + |\hat{a}^{\mathrm{o}}_n|^2)\overset{\times}{\mu}_n \qquad \text{(see (A.69) below)} \\
&= \frac{C_*}{4\sqrt{\underset{+}{g}}_{\min}} (\|a^{\mathrm{e}} + a^{\mathrm{o}}\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h} + \|a^{\mathrm{e}} - a^{\mathrm{o}}\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h}) \\
&= \frac{C_*}{4\sqrt{\underset{+}{g}}_{\min}} \|D_n^- \psi\|^2_{\overset{\times}{H}{}^{\frac{1}{2}}_h(\Gamma_{\mathrm{g}})}.
\end{aligned} \tag{A.68}$$

Here in the second inequality, we have used the estimates $\overset{\times}{\sigma}{}^{\mathrm{e}}_n \le C_* \overset{\times}{\mu}_n$ and $\overset{\times}{\sigma}{}^{\mathrm{o}}_n \le C_* \overset{\times}{\mu}_n$ which will be given in Lemma A.4 below. ∎

LEMMA A.4. *There is a constant* C_* *such that*

$$\overset{\times}{\sigma}{}^{\mathrm{e}}_n \le C_* \overset{\times}{\mu}_n, \qquad \overset{\times}{\sigma}{}^{\mathrm{o}}_n \le C_* \overset{\times}{\mu}_n \tag{A.69}$$

uniformly for all $0 \le n \le N-1$ *and* h *small enough.*

Proof. We will only show that $\overset{\times}{\sigma}{}^{\mathrm{e}}_n \le C_* \overset{\times}{\mu}_n$, or equivalently

$$\sum_{m \in \overset{\times}{\mathbb{M}}_n} \frac{\cos^2(m\pi h)}{\overset{\times}{\kappa}{}^2_{mn}} \le C_* \sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \frac{\tilde{\lambda}_m^2}{\overset{\times}{\kappa}{}^4_{mn}}, \qquad \text{uniformly in } n. \tag{A.70}$$

The proof for $\overset{\times}{\sigma}{}^{\mathrm{o}}_n \le C_* \overset{\times}{\mu}_n$ is similar. Denote by

$$s_{m/2} \triangleq \sin(m\pi h/2), \qquad c_{m/2} \triangleq \cos(m\pi h/2). \tag{A.71}$$

It follows that

$$\text{LHS of (A.70)} = \sum_{m \in \overset{\times}{\mathbb{M}}_n} \frac{c_m^2}{\overset{\times}{\kappa}{}^2_{mn}} \le \sum_{m \in \overset{\times}{\mathbb{M}}_n} \frac{1}{\overset{\times}{\kappa}{}^2_{mn}} = \frac{h^2}{4} \sum_{m \in \overset{\times}{\mathbb{M}}_n} \frac{1}{s^2_{m/2} c_n^2 + s_n^2 c^2_{m/2}} \triangleq (\mathbf{L})_n, \tag{A.72}$$

$$\text{RHS of (A.70)} = \sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \frac{\tilde{\lambda}_m^2}{\overset{\times}{\kappa}{}^4_{mn}} = \frac{h^2}{4} \sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \frac{s^2_{m/2} c^2_{m/2}}{(s^2_{m/2} c_n^2 + s_n^2 c^2_{m/2})^2} \triangleq (\mathbf{R})_n. \tag{A.73}$$

It suffices to show that

$$(\mathbf{L})_n \le C_* (\mathbf{R})_n \tag{A.74}$$

for some constant C_* independent of n and h. We show it separately for the following cases:
Case 1: $1 \le n \le N/2 - 1$ or $N/2 + 1 \le n \le N - 1$.

In this case, $\overset{\times}{\mathbb{M}}_n = \{0, 1, \cdots, N\}$. Since the expressions in (A.72) and (A.73) are symmetric with respect to $n = N/2$, it suffices to consider the case of $1 \le n \le N/2 - 1$. Denote by

$$x_{m/2} = \frac{m\pi h}{2}, \qquad y_n = n\pi h, \tag{A.75}$$

and let

$$f_y(x) \triangleq \frac{1}{\sin^2 x \cos^2 y + \sin^2 y \cos^2 x}, \qquad x \in [0, \frac{\pi}{2}],\ y \in [\pi h, \frac{\pi}{2} - \pi h] \tag{A.76}$$

$$g_y(x) \triangleq \frac{\sin^2 x \cos^2 x}{(\sin^2 x \cos^2 y + \sin^2 y \cos^2 x)^2}, \qquad x \in [0, \frac{\pi}{2}],\ y \in [\pi h, \frac{\pi}{2} - \pi h] \tag{A.77}$$

We can rewrite (A.72) and (A.73) as

$$(\mathbf{L})_n = \frac{h^2}{4} \sum_{m \in \overset{\times}{\mathbb{M}}_n} \frac{1}{s^2_{m/2} c_n^2 + s_n^2 c^2_{m/2}} = \frac{h^2}{4} \sum_{m=0}^{N} f_{y_n}(x_{m/2}), \tag{A.78}$$

$$(\mathbf{R})_n = \frac{h^2}{4} \sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} \frac{s^2_{m/2} c^2_{m/2}}{(s^2_{m/2} c_n^2 + s_n^2 c^2_{m/2})^2} = \frac{h^2}{4} \sum_{\substack{m=1\\ m \text{ is even}}}^{N-1} g_{y_n}(x_{m/2}). \tag{A.79}$$

Since

$$\frac{d}{dx}\Big(\frac{1}{f_y(x)}\Big) = \sin(2x)\cos(2y), \tag{A.80}$$

it follows that $f_y(\cdot)$ is decreasing if $y \in [0, \pi/4]$ and increasing if $y \in [\pi/4, \pi/2]$. In either case, the integral test can be used to estimate the sum in (A.79) to get

$$
(\mathbf{L})_n = \frac{h^2}{4}\sum_{m=0}^{N} f_{y_n}(x_{m/2})
$$

$$
\text{(A.81)} \qquad \leq \begin{cases} \dfrac{h^2}{4} f_{y_n}(x_0) + \dfrac{h}{4}\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} f_{y_n}(x)\,dx = \dfrac{h^2}{4}\dfrac{1}{\sin^2 y_n} + \dfrac{h}{4\sin y_n \cos y_n}, & 1 \leq n \leq \dfrac{N}{4} \\ \dfrac{h^2}{4} f_{y_n}(x_N) + \dfrac{h}{4}\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} f_{y_n}(x)\,dx = \dfrac{h^2}{4}\dfrac{1}{\cos^2 y_n} + \dfrac{h}{4\sin y_n \cos y_n}, & \dfrac{N}{4} \leq n \leq \dfrac{N}{2} - 1 \end{cases}
$$

where we have used the identity $\displaystyle\int_0^{\pi/2} \frac{1}{a^2\sin^2 x + b^2\cos^2 x}dx = \frac{\pi}{2|ab|}$.

On the other hand,

$$
\text{(A.82)} \qquad \frac{d}{dx}\Big(\frac{1}{g_y(x)}\Big) = 2(\tan x\cos^2 y + \cot x \sin^2 y)(\sec^2 x\cos^2 y - \csc^2 x \sin^2 y),
$$

it follows that $g_y(\cdot)$ is increasing when $0 \leq x \leq y$, deceasing when $y \leq x \leq \pi/2$ and therefore attains its maximum $\dfrac{1}{4\sin^2 y\cos^2 y}$ at $x = y$. Consequently,

$$
\text{(A.83)} \qquad \pi h\Big(\sum_{\substack{m=1 \\ m \text{ is even}}}^{N-1} g_{y_n}(x_{m/2}) + \max_{x\in[0,\pi/2]} g_{y_n}(x)\Big) \geq \int_0^{\pi/2} g_{y_n}(x)\,dx = \frac{\pi}{4\sin y_n\cos y_n(\sin y_n + \cos y_n)^2}
$$

where we have used the identity $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}dx = \frac{\pi}{4|ab|(|a|+|b|)^2}$. Therefore from (A.79), we have

$$
\text{(A.84)} \qquad \begin{aligned} (\mathbf{R})_n = \frac{h^2}{4}\sum_{\substack{m=1 \\ m \text{ is even}}}^{N-1} g_{y_n}(x_{m/2}) &\geq \frac{h}{4\pi}\Big(\int_0^{\pi/2} g_{y_n}(x)\,dx - \pi h \max_{x\in[0,\pi/2]} g_{y_n}(x)\Big) \\ &= \frac{h}{16\sin y_n\cos y_n(\sin y_n + \cos y_n)^2} - \frac{h^2}{16\sin^2 y_n\cos^2 y_n}. \end{aligned}
$$

Combining (A.81) and (A.84), we obtain

$$
\text{(A.85)} \qquad \frac{(\mathbf{L})_n}{(\mathbf{R})_n} \leq \begin{cases} \dfrac{4\cos y_n(\sin y_n + \cos y_n)^2(h\cos y_n + \sin y_n)}{\sin y_n\cos y_n - h(\sin y_n + \cos y_n)^2}, & 1 \leq n \leq \frac{N}{4}, \\ \dfrac{4\sin y_n(\sin y_n + \cos y_n)^2(h\sin y_n + \cos y_n)}{\sin y_n\cos y_n - h(\sin y_n + \cos y_n)^2}, & \frac{N}{4} \leq n \leq \frac{N}{2} - 1. \end{cases}
$$

That is,

$$
\text{(A.86)} \qquad \frac{(\mathbf{L})_n}{(\mathbf{R})_n} \leq \begin{cases} \dfrac{4(\sin y_n + \cos y_n)^2(1 + h\cot y_n)}{1 - 2h(\csc(2y_n) + 1)}, & 1 \leq n \leq \dfrac{N}{4}, \\ \dfrac{4(\sin y_n + \cos y_n)^2(1 + h\tan y_n)}{1 - 2h(\csc(2y_n) + 1)}, & \dfrac{N}{4} \leq n \leq \dfrac{N}{2} - 1. \end{cases}
$$

In either case, we have

$$(A.87)\qquad \frac{(\mathbf{L})_n}{(\mathbf{R})_n} \le \begin{cases} \dfrac{8(1+h\cot y_1)}{1-2h(\csc(2y_1)+1)}, & 1 \le n \le \dfrac{N}{4} \\ \dfrac{8(1+h\tan y_{\frac{N}{2}-1})}{1-2h(\csc(2y_{\frac{N}{2}-1})+1)}, & \dfrac{N}{4} \le n \le \dfrac{N}{2}-1 \end{cases} = 8\frac{1+\frac{1}{\pi}}{1-\frac{1}{\pi}} + O(h).$$

This completes the proof of (A.74) for case 1.

Case 2: $n = 0$.

Using the same argument as in the proof of case 1, we have

$$(A.88)\qquad \begin{aligned} (\mathbf{L})_0 = \frac{h^2}{4}\sum_{m=1}^{N}\frac{1}{s_{m/2}^2} &\le \frac{h^2}{4}\Big(\frac{1}{s_{1/2}^2} + \sum_{m=2}^{N}\frac{1}{s_{m/2}^2}\Big) \\ &\le \frac{h^2}{4}\csc^2\left(\frac{\pi h}{2}\right) + \frac{h}{2\pi}\int_{\frac{\pi h}{2}}^{\frac{\pi}{2}}\csc^2\theta\, d\theta \\ &= \frac{h^2}{4}\csc^2(\frac{\pi h}{2}) + \frac{h}{2\pi}\cot(\frac{\pi h}{2}). \end{aligned}$$

and

$$(A.89)\qquad (\mathbf{R})_0 = \frac{h^2}{4}\sum_{\substack{m=1 \\ m \text{ is even}}}^{N-1}\frac{c_{m/2}^2}{s_{m/2}^2} \ge \frac{h}{4\pi}\int_{\pi h}^{\frac{\pi}{2}}\cot^2\theta\, d\theta = \frac{h}{4\pi}\left(\cot(\pi h) + \pi h - \frac{\pi}{2}\right).$$

Hence

$$(A.90)\qquad \frac{(\mathbf{L})_0}{(\mathbf{R})_0} \le \frac{\frac{h^2}{4}\csc^2(\frac{\pi h}{2}) + \frac{h}{2\pi}\cot(\frac{\pi h}{2})}{\frac{h}{4\pi}\left(\cot(\pi h) + \pi h - \frac{\pi}{2}\right)} = 8 + O(h).$$

This completes the proof of (A.74) for case 2.

Case 3: $n = N/2$.

In this case,

$$(A.91)\qquad (\mathbf{L})_{N/2} = \frac{h^2}{4}\sum_{m=0}^{N-1}\frac{1}{c_{m/2}^2}, \qquad (\mathbf{R})_{N/2} = \frac{h^2}{4}\sum_{\substack{m=1 \\ m \text{ is even}}}^{N-1}\frac{s_{m/2}^2}{c_{m/2}^2}$$

The estimate

$$(A.92)\qquad \frac{(\mathbf{L})_{N/2}}{(\mathbf{R})_{N/2}} \le 8 + O(h)$$

follows from the same argument as in case 2.

In view of (A.87, A.90, A.92), the estimate (A.74) follows and the proof for Lemma A.4 is completed. ☐

Denote by

$$(A.93)\qquad \|a\|_0 = \langle a, a\rangle_{0,\bar{\Omega}_g}^{\frac{1}{2}} = \left(h^2 \sum_{i=0}^{N}{}' \sum_{j=1}^{N} a_{ij}^2\right)^{\frac{1}{2}}.$$

The following technical Lemma plays an essential role in the proof of the general case of Theorem 2. It is worth noting that the estimate (A.94) is purely discrete and algebraic. In other words, it is independent of the coordinate mapping.

LEMMA A.5. *There exists a constant $\tilde{C} > 0$, independent of h, such that*

$$\|\overset{\times}{D}'_1\overset{\times}{v}_1 - \overset{\times}{D}'_2\overset{\times}{v}_2\|_0^2 + \|\overset{\times}{D}'_1\overset{\times}{v}_2 + \overset{\times}{D}'_2\overset{\times}{v}_1\|_0^2 \le \tilde{C}\left(\|\overset{\times}{D}'_1\overset{\times}{v}_1 + \overset{\times}{D}'_2\overset{\times}{v}_2\|_0^2 + \|\overset{\times}{D}'_1\overset{\times}{v}_2 - \overset{\times}{D}'_2\overset{\times}{v}_1\|_0^2\right) \tag{A.94}$$

for all $\boldsymbol{v} \in L^2(\underset{+}{\Omega}_{\mathrm{c}}, \mathbb{R}^2)$.

Proof. We first rewrite the expressions in (A.94) in terms of components and difference operators in the default coordinate (ξ^1, ξ^2):

$$\begin{aligned}
\overset{\times}{D}'_1\overset{\times}{v}_1 + \overset{\times}{D}'_2\overset{\times}{v}_2 &= A_2 D'_1 v_1 + A'_1 D_2 v_2,\\
\overset{\times}{D}'_1\overset{\times}{v}_2 - \overset{\times}{D}'_2\overset{\times}{v}_1 &= A_2 D'_1 v_2 - A'_1 D_2 v_1,\\
\overset{\times}{D}'_1\overset{\times}{v}_1 - \overset{\times}{D}'_2\overset{\times}{v}_2 &= A'_1 D_2 v_1 + A_2 D'_1 v_2,\\
\overset{\times}{D}'_1\overset{\times}{v}_2 + \overset{\times}{D}'_2\overset{\times}{v}_1 &= A'_1 D_2 v_2 - A_2 D'_1 v_1,
\end{aligned} \tag{A.95}$$

where the averaging and differencing operators are defined by

$$(A'_1 f)_i = \begin{cases} f_{\frac{1}{2}} & i = 0\\ \frac{1}{2}(f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}) & 1 \le i \le N-1\ ,\\ f_{N-\frac{1}{2}} & i = N \end{cases} \qquad (A_2 g)_j = \frac{1}{2}(g_{j+\frac{1}{2}} + g_{j-\frac{1}{2}}), \quad 1 \le j \le N. \tag{A.96}$$

$$(D'_1 f)_i = \begin{cases} \frac{2}{h} f_{\frac{1}{2}} & i = 0\\ \frac{1}{h}(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}) & 1 \le i \le N-1\ ,\\ \frac{-2}{h} f_{N-\frac{1}{2}} & i = N \end{cases} \qquad (D_2 g)_j = \frac{1}{h}(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}), \quad 1 \le j \le N. \tag{A.97}$$

The reduced averaging and differencing are not needed in ξ^2 due to periodic assumption. Next, we decompose the quantities in (A.94) into

$$\|\overset{\times}{D}'_1\overset{\times}{v}_1 + \overset{\times}{D}'_2\overset{\times}{v}_2\|_0^2 + \|\overset{\times}{D}'_1\overset{\times}{v}_2 - \overset{\times}{D}'_2\overset{\times}{v}_1\|_0^2 = \text{(I)} + \text{(II)} + \text{(III)}, \tag{A.98}$$

$$\|\overset{\times}{D}'_1\overset{\times}{v}_1 - \overset{\times}{D}'_2\overset{\times}{v}_2\|_0^2 + \|\overset{\times}{D}'_1\overset{\times}{v}_2 + \overset{\times}{D}'_2\overset{\times}{v}_1\|_0^2 = \text{(I)} + \text{(II)} - \text{(III)}, \tag{A.99}$$

where

$$\text{(I)} \triangleq h^2 {\sum_{i=0}^{N}}' \sum_{j=1}^{N} \left((A_2 D'_1 v_1)^2 + (A_2 D'_1 v_2)^2\right)_{i,j} + h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N} \left((A_1 D_2 v_1)^2 + (A_1 D_2 v_2)^2\right)_{i,j}, \tag{A.100}$$

$$\text{(II)} \triangleq \frac{h^2}{2} \sum_{j=1}^{N} \left((D_2 v_1)^2 + (D_2 v_2)^2\right)_{\frac{1}{2},j} + \frac{h^2}{2} \sum_{j=1}^{N} \left((D_2 v_1)^2 + (D_2 v_2)^2\right)_{N-\frac{1}{2},j}, \tag{A.101}$$

$$\text{(III)} \triangleq 2h \sum_{j=1}^{N} (v_1 \tilde{D}_2 v_2)_{\frac{1}{2},j+\frac{1}{2}} - 2h \sum_{j=1}^{N} (v_1 \tilde{D}_2 v_2)_{N-\frac{1}{2},j+\frac{1}{2}}, \tag{A.102}$$

and $\tilde{D}_2$ is the long-stencil centered difference in ξ^2 direction,

$$(\tilde{D}_2 v_2)_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{(v_2)_{i-\frac{1}{2},j+\frac{1}{2}} - (v_2)_{i-\frac{1}{2},j-\frac{3}{2}}}{2h}. \tag{A.103}$$

We will show that

$$|(\mathrm{III})| \le \theta\big((\mathrm{I}) + (\mathrm{II})\big) \quad \text{for some fixed constant } \theta \in (0,1). \tag{A.104}$$

Then (A.94) follows as a consequence.

We first define

$$\mathbb{S}^m_{i-\frac{1}{2}} \triangleq \begin{cases} \sqrt{2}\sin(m\pi\xi^1_{i-\frac{1}{2}}) & 1 \le m \le N-1; \\ \sin(m\pi\xi^1_{i-\frac{1}{2}}) & m = N, \end{cases} \tag{A.105}$$

$$\mathbb{E}^n_{j-\frac{1}{2}} \triangleq \exp(2n\pi\sqrt{-1}\xi^2_{j-\frac{1}{2}}) \qquad 0 \le n \le N-1. \tag{A.106}$$

Thus

$$\{\mathbb{S}^m \otimes \mathbb{E}^n \,|\, 1 \le m \le N,\ 0 \le n \le N-1.\} \tag{A.107}$$

is an orthonormal basis of $L^2(\underset{+}{\Omega}_c, \mathbb{R})$ with respect to the inner product

$$\langle a, b\rangle_{0,\underset{+}{\Omega}_c} = h^2 \sum_{i=1}^{N}\sum_{j=1}^{N} (ab)_{i-\frac{1}{2},j-\frac{1}{2}}. \tag{A.108}$$

We can now expand the components of $\boldsymbol{v}$ with respect to the basis (A.107):

$$(v_1)_{i-\frac{1}{2},j-\frac{1}{2}} = \sum_{m=1}^{N}\sum_{n=0}^{N-1} a_{mn}\mathbb{S}^m_{i-\frac{1}{2}}\mathbb{E}^n_{j-\frac{1}{2}}, \quad 1 \le i \le N,\ 1 \le j \le N; \tag{A.109}$$

$$(v_2)_{i-\frac{1}{2},j-\frac{1}{2}} = \sum_{p=1}^{N}\sum_{n=0}^{N-1} b_{pn}\mathbb{S}^p_{i-\frac{1}{2}}\mathbb{E}^n_{j-\frac{1}{2}}, \quad 1 \le i \le N,\ 1 \le j \le N. \tag{A.110}$$

It follows that

$$(D_1' v_1)_{i,j-\frac{1}{2}} = \sum_{m=1}^{N}\sum_{n=0}^{N-1} \lambda_m a_{mn}\mathbf{C}^m_i\mathbb{E}^n_{j-\frac{1}{2}}, \quad 0 \le i \le N,\ 1 \le j \le N; \tag{A.111}$$

$$(D_2 v_1)_{i-\frac{1}{2},j} = \sqrt{-1}\sum_{m=1}^{N}\sum_{n=0}^{N-1} \lambda_{2n} a_{mn}\mathbb{S}^m_{i-\frac{1}{2}}\mathbf{E}^n_j, \quad 1 \le i \le N,\ 1 \le j \le N. \tag{A.112}$$

Similar for $D_1' v_2$ and $D_2 v_2$. Moreover,

$$(\tilde{D}_2 v_2)_{i-\frac{1}{2},j-\frac{1}{2}} = \sqrt{-1}\sum_{p=1}^{N}\sum_{n=0}^{N-1} \tilde{\lambda}_{2n} b_{pn}\mathbb{S}^p_{i-\frac{1}{2}}\mathbb{E}^n_{j-\frac{1}{2}}, \quad i = 1, N,\ 1 \le j \le N; \tag{A.113}$$

where

$$\begin{cases} \dfrac{\mathbb{S}^m_{i+\frac{1}{2}} + \mathbb{S}^m_{i-\frac{1}{2}}}{2} = c_{m/2}\mathbf{S}^m_i, & 1 \le i \le N-1, \quad 1 \le m \le N-1; \\ \dfrac{\mathbb{S}^N_{i+\frac{1}{2}} + \mathbb{S}^N_{i-\frac{1}{2}}}{2} = 0, & 1 \le i \le N-1; \\ \dfrac{\mathbb{E}^n_{j+\frac{1}{2}} + \mathbb{E}^n_{j-\frac{1}{2}}}{2} = c_n\mathbf{E}^n_j, & 1 \le j \le N, \quad 0 \le n \le N-1. \end{cases} \tag{A.114}$$

and $c_{m/2} = \cos(m\pi h/2)$, $c_n = \cos(n\pi h)$. Therefore

$$
\text{(A.115)}\quad \begin{aligned}
\text{(I)} &= h^2 \sum_{i=0}^{N}{}' \sum_{j=1}^{N} \big((A_2 D_1' v_1)^2 + (A_2 D_1' v_2)^2\big)_{i,j} + h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N} \big((A_1 D_2 v_1)^2 + (A_1 D_2 v_2)^2\big)_{i,j} \\
&= \frac{4}{h^2} \sum_{m=1}^{N} \sum_{n=0}^{N-1} c_n^2 s_{m/2}^2 (|a_{mn}|^2 + |b_{mn}|^2) + \sum_{m=1}^{N-1} \sum_{n=0}^{N-1} s_n^2 c_{m/2}^2 (|a_{mn}|^2 + |b_{mn}|^2)
\end{aligned}
$$

where $s_{m/2} = \sin(m\pi h/2)$, $s_n = \sin(n\pi h)$. Since $s_n = \frac{h}{2}\lambda_{2n}$, we also have

$$
\text{(A.116)}\quad \begin{aligned}
\text{(II)}_{\frac12} &\triangleq \frac{h^2}{2} \sum_{j=1}^{N} \big((D_2 v_1)^2 + (D_2 v_2)^2\big)_{\frac12, j} \\
&= \frac{h}{2} \sum_{m,k=1}^{N} \sum_{n=0}^{N-1} a_{mn} a_{kn}^* \lambda_{2n}^2 \mathbb{S}_{\frac12}^m \mathbb{S}_{\frac12}^k + \frac{h}{2} \sum_{p,r=1}^{N} \sum_{n=0}^{N-1} b_{pn} b_{rn}^* \lambda_{2n}^2 \mathbb{S}_{\frac12}^p \mathbb{S}_{\frac12}^r \\
&= \frac{2}{h} \sum_{n=0}^{N-1} s_n^2 \Big(\big(\sum_{m=1}^{N} a_{mn} \mathbb{S}_{\frac12}^m\big)\big(\sum_{m=1}^{N} a_{mn} \mathbb{S}_{\frac12}^m\big)^* + \big(\sum_{p=1}^{N} b_{pn} \mathbb{S}_{\frac12}^p\big)\big(\sum_{p=1}^{N} b_{pn} \mathbb{S}_{\frac12}^p\big)^* \Big),
\end{aligned}
$$

$$
\text{(A.117)}\quad \begin{aligned}
\text{(II)}_{N-\frac12} &\triangleq \frac{h^2}{2} \sum_{j=1}^{N} \big((D_2 v_1)^2 + (D_2 v_2)^2\big)_{N-\frac12, j} \\
&= \frac{h}{2} \sum_{m,k=1}^{N} \sum_{n=0}^{N-1} a_{mn} a_{kn}^* \lambda_{2n}^2 \mathbb{S}_{\frac12}^m \mathbb{S}_{N-\frac12}^k + \frac{h}{2} \sum_{p,r=1}^{N} \sum_{n=0}^{N-1} b_{pn} b_{rn}^* \lambda_{2n}^2 \mathbb{S}_{\frac12}^p \mathbb{S}_{N-\frac12}^r \\
&= \frac{2}{h} \sum_{n=0}^{N-1} s_n^2 \Big(\big(\sum_{m=1}^{N} a_{mn} \mathbb{S}_{N-\frac12}^m\big)\big(\sum_{m=1}^{N} a_{mn} \mathbb{S}_{N-\frac12}^m\big)^* + \big(\sum_{p=1}^{N} b_{pn} \mathbb{S}_{N-\frac12}^p\big)\big(\sum_{p=1}^{N} b_{pn} \mathbb{S}_{N-\frac12}^p\big)^* \Big).
\end{aligned}
$$

Since $\mathbb{S}_{N-\frac12}^m = (-1)^{m+1} \mathbb{S}_{\frac12}^m$, we can write

$$
\text{(A.118)}\quad \sum_{m=1}^{N} a_{mn} \mathbb{S}_{\frac12}^m = \sum_{\substack{m=1 \\ m=\text{odd}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m + \sum_{\substack{m=1 \\ m=\text{even}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m,
$$

$$
\text{(A.119)}\quad \sum_{m=1}^{N} a_{mn} \mathbb{S}_{N-\frac12}^m = \sum_{\substack{m=1 \\ m=\text{odd}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m - \sum_{\substack{m=1 \\ m=\text{even}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m.
$$

It follows that

$$
\text{(A.120)}\quad \begin{aligned}
\text{(II)} &= \text{(II)}_{\frac12} + \text{(II)}_{N-\frac12} \\
&= \frac{4}{h} \sum_{n=0}^{N-1} s_n^2 \Big(\big|\sum_{\substack{m=1 \\ m=\text{odd}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m\big|^2 + \big|\sum_{\substack{m=1 \\ m=\text{even}}}^{N} a_{mn} \mathbb{S}_{\frac12}^m\big|^2 + \big|\sum_{\substack{p=1 \\ p=\text{odd}}}^{N} b_{pn} \mathbb{S}_{\frac12}^p\big|^2 + \big|\sum_{\substack{p=1 \\ p=\text{even}}}^{N} b_{pn} \mathbb{S}_{\frac12}^p\big|^2 \Big)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\mathrm{III}) &= 2h\sum_{j=1}^{N}(v_1\tilde{D}_2v_2)_{\frac{1}{2},j-\frac{1}{2}} - 2h\sum_{j=1}^{N}(v_1\tilde{D}_2v_2)_{N-\frac{1}{2},j-\frac{1}{2}} \\
&= -4\sqrt{-1}\sum_{\substack{m,p=1\\ m+p=\text{odd}}}^{N}\Big(\sum_{n=1}^{N}a_{mn}b_{pn}^{*}\lambda_{2n}c_n\Big)\mathbb{S}_{\frac{1}{2}}^{m}\mathbb{S}_{\frac{1}{2}}^{p} \\
&= \frac{-8\sqrt{-1}}{h}\sum_{n=0}^{N-1}s_nc_n\Big(\big(\sum_{\substack{m=1\\ m=\text{even}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{odd}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*} + \big(\sum_{\substack{m=1\\ m=\text{odd}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{even}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*}\Big).
\end{aligned}
\tag{A.121}
$$

In summary, we have

$$
(\mathrm{I}) =\frac{4}{h^2}\sum_{n=0}^{N-1}\Big(c_n^2\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big)+s_n^2\sum_{m=1}^{N-1}\big((|a_{mn}|^2+|b_{mn}|^2)c_{m/2}^2\big)\Big),
\tag{A.122}
$$

$$
(\mathrm{II}) =\frac{4}{h}\sum_{n=0}^{N-1}s_n^2\Big(\big|\sum_{\substack{m=1\\ m=\text{odd}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big|^2+\big|\sum_{\substack{m=1\\ m=\text{even}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big|^2+\big|\sum_{\substack{p=1\\ p=\text{odd}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big|^2+\big|\sum_{\substack{p=1\\ p=\text{even}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big|^2\Big),
\tag{A.123}
$$

$$
\begin{aligned}
(\mathrm{III}) &=-4\sqrt{-1}\sum_{\substack{m,p=1\\ m+p=\text{odd}}}^{N}\Big(\sum_{n=0}^{N-1}a_{mn}b_{pn}^{*}\lambda_{2n}c_n\Big)\mathbb{S}_{\frac{1}{2}}^{m}\mathbb{S}_{\frac{1}{2}}^{p} \\
&=\frac{-8\sqrt{-1}}{h}\sum_{n=0}^{N-1}s_nc_n\Big(\big(\sum_{\substack{m=1\\ m=\text{even}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{odd}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*} + \big(\sum_{\substack{m=1\\ m=\text{odd}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{even}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*}\Big).
\end{aligned}
\tag{A.124}
$$

We now decompose

$$
(\mathrm{I}) = \sum_{n=0}^{N-1}(\mathrm{I})_n, \quad (\mathrm{II}) = \sum_{n=0}^{N-1}(\mathrm{II})_n, \quad (\mathrm{III}) = \sum_{n=0}^{N-1}(\mathrm{III})_n,
\tag{A.125}
$$

where

$$
(\mathrm{I})_n =\frac{4}{h^2}\Big(c_n^2\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big)+s_n^2\sum_{m=1}^{N-1}\big((|a_{mn}|^2+|b_{mn}|^2)c_{m/2}^2\big)\Big),
\tag{A.126}
$$

$$
(\mathrm{II})_n =\frac{4}{h}s_n^2\Big(\big|\sum_{\substack{m=1\\ m=\text{odd}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big|^2+\big|\sum_{\substack{m=1\\ m=\text{even}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big|^2+\big|\sum_{\substack{p=1\\ p=\text{odd}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big|^2+\big|\sum_{\substack{p=1\\ p=\text{even}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big|^2\Big),
\tag{A.127}
$$

$$
\begin{aligned}
(\mathrm{III})_n &=-4\sqrt{-1}\sum_{\substack{m,p=1\\ m+p=\text{odd}}}^{N}a_{mn}b_{pn}^{*}\lambda_{2n}c_n\mathbb{S}_{\frac{1}{2}}^{m}\mathbb{S}_{\frac{1}{2}}^{p} \\
&=\frac{-8\sqrt{-1}}{h}s_nc_n\Big(\big(\sum_{\substack{m=1\\ m=\text{even}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{odd}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*} + \big(\sum_{\substack{m=1\\ m=\text{odd}}}^{N}a_{mn}\mathbb{S}_{\frac{1}{2}}^{m}\big)\big(\sum_{\substack{p=1\\ p=\text{even}}}^{N}b_{pn}\mathbb{S}_{\frac{1}{2}}^{p}\big)^{*}\Big),
\end{aligned}
\tag{A.128}
$$

and proceed to show that

$$|(\mathrm{III})_n| \le \theta\big((\mathrm{I})_n + (\mathrm{II})_n\big), \quad \theta \in (0,1), \tag{A.129}$$

for the following two cases separately.
Case 1: $c_n^2 \le \frac{9}{25}$.

In this case, it is clear from (A.127), (A.128) and Cauchy-Schwartz inequality that

$$|(\mathrm{III})_n| \le \frac{|c_n|}{s_n}(\mathrm{II})_n \le \frac{3}{4}\big((\mathrm{I})_n + (\mathrm{II})_n\big). \tag{A.130}$$

Case 2: $c_n^2 \ge \frac{9}{25}$.

We can write

$$\begin{aligned}
|(\mathrm{III})_n| &= \Big| -4\sqrt{-1}\sum_{\substack{m,p=1\\ m+p=\mathrm{odd}}}^{N} a_{mn}b_{pn}^{*}\lambda_{2n}c_n\mathbb{S}_{\frac{1}{2}}^{m}\mathbb{S}_{\frac{1}{2}}^{p}\Big| \\
&\le 2\sum_{\substack{m,p=1\\ m+p=\mathrm{odd}}}^{N} \big(|a_{mn}|^2 + |b_{pn}|^2\big)\lambda_{2n}|c_n|\mathbb{S}_{\frac{1}{2}}^{m}\mathbb{S}_{\frac{1}{2}}^{p} \\
&= 2\sum_{m=1}^{N}|a_{mn}|^2\lambda_{2n}|c_n|\frac{\mathbb{S}_{\frac{1}{2}}^{m}}{h}\Big(h\sum_{\substack{p=1\\ m+p=\mathrm{odd}}}^{N}\mathbb{S}_{\frac{1}{2}}^{p}\Big) + 2\sum_{p=1}^{N}|b_{pn}|^2\lambda_{2n}|c_n|\frac{\mathbb{S}_{\frac{1}{2}}^{p}}{h}\Big(h\sum_{\substack{m=1\\ m+p=\mathrm{odd}}}^{N}\mathbb{S}_{\frac{1}{2}}^{m}\Big).
\end{aligned} \tag{A.131}$$

Note that from (A.105),

$$h\sum_{\substack{p=1\\ m+p=\mathrm{odd}}}^{N}\mathbb{S}_{\frac{1}{2}}^{p} = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\sin x\,dx + O(h) = \frac{\sqrt{2}}{\pi} + O(h). \tag{A.132}$$

and

$$\lambda_m = \frac{2\sin(m\pi h/2)}{h} = \begin{cases} \dfrac{\sqrt{2}\mathbb{S}_{\frac{1}{2}}^{m}}{h}, & 1 \le m \le N-1; \\ \dfrac{2\mathbb{S}_{\frac{1}{2}}^{N}}{h}, & m = N. \end{cases} \tag{A.133}$$

Therefore, we obtain from (A.131), (A.132) and (A.133) that

$$\begin{aligned}
|(\mathrm{III})_n| &\le \frac{2}{\pi}\big(1+O(h)\big)\Big(\sum_{m=1}^{N}|a_{mn}|^2\lambda_{2n}|c_n|\lambda_m + \sum_{p=1}^{N}|b_{pn}|^2\lambda_{2n}|c_n|\lambda_p\Big) \\
&\le \big(\frac{1}{\pi}+O(h)\big)\Big(\sum_{m=1}^{N}|a_{mn}|^2(\lambda_{2n}^2 + |c_n\lambda_m|^2) + \sum_{p=1}^{N}|b_{pn}|^2(\lambda_{2n}^2 + |c_n\lambda_p|^2)\Big) \\
&= \big(\frac{1}{\pi}+O(h)\big)\frac{4}{h^2}\Big(c_n^2\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big) + s_n^2\sum_{m=1}^{N}\big(|a_{mn}|^2+|b_{mn}|^2\big)\Big).
\end{aligned} \tag{A.134}$$

On the other hand, from (A.126), we have

$$
\begin{aligned}
(\mathrm{I})_n &\geq \frac{4}{h^2}\Big((c_n^2 - c_n^2 s_n^2 + c_n^2 s_n^2)\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big) + c_n^2 s_n^2 \sum_{m=1}^{N-1}\big((|a_{mn}|^2+|b_{mn}|^2)c_{m/2}^2\big)\Big) \\
&= \frac{4}{h^2}\Big((c_n^2 - c_n^2 s_n^2)\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big) + c_n^2 s_n^2 \sum_{m=1}^{N}\big(|a_{mn}|^2+|b_{mn}|^2\big)\Big) \\
&= c_n^2\frac{4}{h^2}\Big(c_n^2\sum_{m=1}^{N}\big((|a_{mn}|^2+|b_{mn}|^2)s_{m/2}^2\big) + s_n^2 \sum_{m=1}^{N}\big(|a_{mn}|^2+|b_{mn}|^2\big)\Big).
\end{aligned}
\tag{A.135}
$$

Therefore, since $c_n^2 \geq \frac{9}{25}$,

$$
|(\mathrm{III})_n| \leq \big(\frac{1}{\pi}+O(h)\big)\frac{(\mathrm{I})_n}{c_n^2} \leq \big(\frac{25}{9\pi}+O(h)\big)(\mathrm{I})_n \leq \frac{25}{27}\big((\mathrm{I})_n + (\mathrm{II})_n\big).
\tag{A.136}
$$

The estimate (A.129) and therefore (A.104) then follows from (A.130) and (A.136) with $\theta = \frac{25}{27}$.

In view of (A.98), (A.99) and (A.104), we have proved (A.94) with $\tilde{C} = \dfrac{1+\theta}{1-\theta} = 26$. ☐

A.2. The LBB Condition for General Lifshitz Metrics. We now proceed to prove the LBB condition for the general case where the coordinate mapping $\boldsymbol{x}$ is a $C^{1,1}$ bijection.

Given $p \in L^2(\bar{\Omega}_\mathrm{g}, \mathbb{R})/\mathbb{R}^2$, we will construct the corresponding vector field $\boldsymbol{u}$, based on the contra-variant components of $\boldsymbol{v}$ given in Theorem 5, such that

$$
\frac{\langle\, p\,,\, \overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u}\,\rangle_{\bar{\Omega}_\mathrm{g}}}{\|p\|_{\bar{\Omega}_\mathrm{g}}\|\boldsymbol{u}\|_{\overset{\times}{H}{}^1_h}} \geq \beta \quad \text{uniformly in } h.
\tag{A.137}
$$

Denote by $I : L^2(\bar{\Omega}_\mathrm{g}, \mathbb{R}) \mapsto L^2(\underset{+}{\bar{\Omega}}_\mathrm{g}, \mathbb{R})$ the natural correspondence given by $I(p)_{i,j} = p_{i,j}$, or more precisely $I(p)\big(\underset{+}{\boldsymbol{x}}(\xi_i^1, \xi_j^2)\big) = p\big(\boldsymbol{x}(\xi_i^1,\xi_j^2)\big)$. If $p \in L^2(\bar{\Omega}_\mathrm{g},\mathbb{R})/\mathbb{R}^2$, then $I(p)$ is in $L^2(\underset{+}{\bar{\Omega}}_\mathrm{g},\mathbb{R})$, but not necessarily in $L^2(\underset{+}{\bar{\Omega}}_\mathrm{g},\mathbb{R})/\mathbb{R}^2$. Let q be the projection of $I(p)$ onto $L^2(\underset{+}{\bar{\Omega}}_\mathrm{g},\mathbb{R})/\mathbb{R}^2$:

$$
q = I(p) - \frac{\langle\, I(p)\,,\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{ge}}\,\rangle_{\underset{+}{\bar{\Omega}}_\mathrm{g}}}{\langle\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{ge}}\,,\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{ge}}\,\rangle_{\underset{+}{\bar{\Omega}}_\mathrm{g}}}\mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{ge}} - \frac{\langle\, I(p)\,,\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{go}}\,\rangle_{\underset{+}{\bar{\Omega}}_\mathrm{g}}}{\langle\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{go}}\,,\, \mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{go}}\,\rangle_{\underset{+}{\bar{\Omega}}_\mathrm{g}}}\mathbf{1}_{\underset{+}{\bar{\Omega}}_\mathrm{go}}
\tag{A.138}
$$

Since $p \in L^2(\bar{\Omega}_\mathrm{g},\mathbb{R})/\mathbb{R}^2$ and $I^{-1}(q) - p \in \operatorname{span}\{\mathbf{1}_{\bar{\Omega}_\mathrm{go}}, \mathbf{1}_{\bar{\Omega}_\mathrm{go}}\}$, it follows from (2.17) that $\langle\, p\,,\, I^{-1}(q) - p\,\rangle_{\bar{\Omega}_\mathrm{g}} = 0$. Therefore

$$
\|p\|_{\bar{\Omega}_\mathrm{g}} \leq \|I^{-1}(q)\|_{\bar{\Omega}_\mathrm{g}}
\tag{A.139}
$$

and

$$
\|q\|^2_{\underset{+}{\bar{\Omega}}_\mathrm{g}} \geq \frac{\sqrt{\underset{+}{g}}_{\min}}{\sqrt{g}_{\max}}\|I^{-1}(q)\|^2_{\bar{\Omega}_\mathrm{g}} \geq \frac{\sqrt{\underset{+}{g}}_{\min}}{\sqrt{g}_{\max}}\|p\|^2_{\bar{\Omega}_\mathrm{g}}
\tag{A.140}
$$

From Theorem 5, there exists

$$
\boldsymbol{v} = \underset{+}{\overset{\times}{v}}{}^1 \underset{+}{\overset{\times}{\boldsymbol{e}}}_1 + \underset{+}{\overset{\times}{v}}{}^2 \underset{+}{\overset{\times}{\boldsymbol{e}}}_2 \in L^2(\underset{+}{\Omega}_\mathrm{c}, \mathbb{R}^2)
\tag{A.141}
$$

such that (A.2) and (A.3) hold. We can now construct explicitly $\boldsymbol{u} \in L^2(\Omega_{\mathrm{c}}, \mathbb{R}^2)$ in terms of the contra-variant components of $\boldsymbol{v}$ in (A.141):

$$\boldsymbol{u} \triangleq \overset{\times}{u}{}^1 \overset{\times}{\boldsymbol{e}}_1 + \overset{\times}{u}{}^2 \overset{\times}{\boldsymbol{e}}_2, \quad \text{where } \overset{\times}{u}{}^1 = \frac{\sqrt{\underset{+}{\overset{\times}{g}}}}{\sqrt{\overset{\times}{g}}} \underset{+}{\overset{\times}{v}}{}^1, \ \overset{\times}{u}{}^2 = \frac{\sqrt{\underset{+}{\overset{\times}{g}}}}{\sqrt{\overset{\times}{g}}} \underset{+}{\overset{\times}{v}}{}^2, \tag{A.142}$$

and claim that $\boldsymbol{u}$ satisfies (A.137). To this end, we first note that, by construction

$$\big(\sqrt{\overset{\times}{g}}\overset{\times}{\nabla}{}'_h \cdot \boldsymbol{u}\big)_{ij} = \big(\overset{\times}{D}{}'_1(\sqrt{\underset{+}{\overset{\times}{g}}}\underset{+}{\overset{\times}{v}}{}^1) + \overset{\times}{D}{}'_2(\sqrt{\underset{+}{\overset{\times}{g}}}\underset{+}{\overset{\times}{v}}{}^2)\big)_{ij} = \big(\sqrt{\underset{+}{\overset{\times}{g}}}\underset{+}{\overset{\times}{\nabla}}{}'_h \cdot \boldsymbol{v}\big)_{ij}, \quad 0 \le i \le N, \ 1 \le j \le N. \tag{A.143}$$

Secondly, we have the following crucial estimate:

LEMMA A.6. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be defined by* (A.141) *and* (A.142)*, then*

$$\|\boldsymbol{u}\|_{\overset{\times}{H}{}^1_h} \le C^\dagger \|\boldsymbol{v}\|_{\underset{+}{\overset{\times}{H}}{}^1_h}. \tag{A.144}$$

where $C^\dagger$ is a constant that only depends on $\|\sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{\alpha\beta}\|_{C^{0,1}([0,1]\times S^1)}$, $\sqrt{g}_{\min}$ and $\sqrt{\underset{+}{g}}_{\max}$.

Proof. We start with the estimate for

$$\|\boldsymbol{u}\|^2_{\Omega_{\mathrm{c}}} = h^2 \sum_{i=0}^{N-1}\sum_{j=1}^{N} \big(\sqrt{\overset{\times}{g}}(\overset{\times}{u}{}^1\overset{\times}{u}_1 + \overset{\times}{u}{}^2\overset{\times}{u}_2)\big)_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{A.145}$$

From (A.142), (A.1) and (2.7), we have

$$\sqrt{\overset{\times}{g}}\overset{\times}{u}{}^\alpha = \sqrt{\underset{+}{\overset{\times}{g}}}\underset{+}{\overset{\times}{v}}{}^\alpha = \underset{+}{\overset{\times}{v}}_\alpha, \quad \alpha = 1, 2. \tag{A.146}$$

It follows from (2.4) and (2.6) that

$$\overset{\times}{u}_1 = \overset{\times}{g}_{11}\overset{\times}{u}{}^1 + \overset{\times}{g}_{12}\overset{\times}{u}{}^2 = \overset{\times}{g}_{11}\frac{\sqrt{\underset{+}{\overset{\times}{g}}}}{\sqrt{\overset{\times}{g}}}\underset{+}{\overset{\times}{v}}{}^1 + \overset{\times}{g}_{12}\frac{\sqrt{\underset{+}{\overset{\times}{g}}}}{\sqrt{\overset{\times}{g}}}\underset{+}{\overset{\times}{v}}{}^2 = \overset{\times}{q}{}^{22}\underset{+}{\overset{\times}{v}}_1 - \overset{\times}{q}{}^{12}\underset{+}{\overset{\times}{v}}_2, \tag{A.147}$$

where $\overset{\times}{q}{}^{\alpha\beta} = \sqrt{\overset{\times}{g}}\overset{\times}{g}{}^{\alpha\beta}$. Similarly,

$$\overset{\times}{u}_2 = -\overset{\times}{q}{}^{12}\underset{+}{\overset{\times}{v}}_1 + \overset{\times}{q}{}^{11}\underset{+}{\overset{\times}{v}}_2. \tag{A.148}$$

Hence

$$\sqrt{\overset{\times}{g}}(\overset{\times}{u}{}^1\overset{\times}{u}_1 + \overset{\times}{u}{}^2\overset{\times}{u}_2) = \underset{+}{\overset{\times}{v}}_1\overset{\times}{u}_1 + \underset{+}{\overset{\times}{v}}_2\overset{\times}{u}_2 = \overset{\times}{q}{}^{22}(\underset{+}{\overset{\times}{v}}_1)^2 - 2\overset{\times}{q}{}^{12}\underset{+}{\overset{\times}{v}}_1\underset{+}{\overset{\times}{v}}_2 + \overset{\times}{q}{}^{11}(\underset{+}{\overset{\times}{v}}_2)^2 \le (\overset{\times}{q}{}^{11} + \overset{\times}{q}{}^{22})\big((\underset{+}{\overset{\times}{v}}_1)^2 + (\underset{+}{\overset{\times}{v}}_2)^2\big). \tag{A.149}$$

Moreover, from (A.146),

$$\|\boldsymbol{v}\|^2_{\underset{+}{\Omega}_{\mathrm{c}}} = h^2 \sum_{i=0}^{N-1}\sum_{j=1}^{N} \Big(\sqrt{\underset{+}{\overset{\times}{g}}}(\underset{+}{\overset{\times}{v}}{}^1\underset{+}{\overset{\times}{v}}_1 + \underset{+}{\overset{\times}{v}}{}^2\underset{+}{\overset{\times}{v}}_2)\Big)_{i+\frac{1}{2},j+\frac{1}{2}} = h^2 \sum_{i=0}^{N-1}\sum_{j=1}^{N} \Big((\underset{+}{\overset{\times}{v}}_1)^2 + (\underset{+}{\overset{\times}{v}}_2)^2\Big)_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{A.150}$$

it follows that

$$\|\boldsymbol{u}\|^2_{\Omega_c} \le \|\overset{\times}{q}{}^{11} + \overset{\times}{q}{}^{22}\|_{C^0([0,1]\times S^1)}\|\boldsymbol{v}\|^2_{\underset{+}{\Omega}_{\mathrm{c}}}. \tag{A.151}$$

Next, we proceed with the estimate for $\|\overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u}\|_{\bar{\Omega}_{\rm g}}$.

From (2.21), (2.22) and (A.146), we can write

$$\sqrt{\overset{\times}{g}}\,\overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u} = \overset{\times}{D}{}'_1\overset{\times}{u}_2 - \overset{\times}{D}{}'_2\overset{\times}{u}_1 = \overset{\times}{D}{}'_1\big(-\overset{\times}{q}{}^{21}\underset{+}{\overset{\times}{v}}{}_1 + \overset{\times}{q}{}^{11}\underset{+}{\overset{\times}{v}}{}_2\big) - \overset{\times}{D}{}'_2\big(\overset{\times}{q}{}^{22}\underset{+}{\overset{\times}{v}}{}_1 - \overset{\times}{q}{}^{12}\underset{+}{\overset{\times}{v}}{}_2\big) \tag{A.152}$$

and

$$\sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}'_h\cdot\boldsymbol{v} = \overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_1 + \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_2, \qquad \sqrt{\underset{+}{g}}\,\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot\boldsymbol{v} = \overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_2 - \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_1. \tag{A.153}$$

We then apply the identity

$$D'(fv)_i = \begin{cases} (Af)_i\,(Dv)_i + (Df)_i\,(Av)_i, & 1\le i\le N-1;\\ (A'f)_i\,(D'v)_i, & i=0,N,\end{cases} \tag{A.154}$$

and recast (A.152) into

$$\sqrt{\overset{\times}{g}}\,\overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u} = \boldsymbol{V}_0 + \boldsymbol{V}_1 \tag{A.155}$$

where

$$\boldsymbol{V}_0 \triangleq \begin{cases} -(\overset{\times}{D}_1\overset{\times}{q}{}^{12})(\overset{\times}{A}_1\underset{+}{\overset{\times}{v}}{}_1) + (\overset{\times}{D}_1\overset{\times}{q}{}^{11})(\overset{\times}{A}_1\underset{+}{\overset{\times}{v}}{}_2) - (\overset{\times}{D}_2\overset{\times}{q}{}^{22})(\overset{\times}{A}_2\underset{+}{\overset{\times}{v}}{}_1) + (\overset{\times}{D}_2\overset{\times}{q}{}^{12})(\overset{\times}{A}_2\underset{+}{\overset{\times}{v}}{}_2) & \text{on } \mathring{\Omega}_{\rm g},\\ 0 & \text{on } \Gamma_{\rm g},\end{cases} \tag{A.156}$$

$$\boldsymbol{V}_1 \triangleq -(\overset{\times}{A}{}'_1\overset{\times}{q}{}^{12})(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_1) + (\overset{\times}{A}{}'_1\overset{\times}{q}{}^{11})(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_2) - (\overset{\times}{A}{}'_2\overset{\times}{q}{}^{22})(\overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_1) + (\overset{\times}{A}{}'_2\overset{\times}{q}{}^{12})(\overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_2) \qquad \text{on } \bar{\Omega}_{\rm g}. \tag{A.157}$$

From (A.150), we have the following estimate in terms of Lifchitz norm of the metric tensor:

$$\|\boldsymbol{V}_0\|_0^2 \le 8\big(\max_{\alpha,\beta=1,2}|\overset{\times}{q}{}^{\alpha\beta}|^2_{C^{0,1}([0,1]\times S^1)}\big)\|\boldsymbol{v}\|^2_{\underset{+}{\Omega}_{\rm c}}. \tag{A.158}$$

Furthermore

$$\begin{aligned}\boldsymbol{V}_1 = {} & \frac{-\overset{\times}{A}{}'_1\overset{\times}{q}{}^{12} + \overset{\times}{A}{}'_2\overset{\times}{q}{}^{12}}{2}\big(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_1 + \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_2\big) + \frac{\overset{\times}{A}{}'_1\overset{\times}{q}{}^{11} + \overset{\times}{A}{}'_2\overset{\times}{q}{}^{22}}{2}\big(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_2 - \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_1\big)\\ & + \frac{-\overset{\times}{A}{}'_1\overset{\times}{q}{}^{12} - \overset{\times}{A}{}'_2\overset{\times}{q}{}^{12}}{2}\big(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_1 - \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_2\big) + \frac{\overset{\times}{A}{}'_1\overset{\times}{q}{}^{11} - \overset{\times}{A}{}'_2\overset{\times}{q}{}^{22}}{2}\big(\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_2 + \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_1\big).\end{aligned} \tag{A.159}$$

It follows from (A.159) and Lemma A.5 that,

$$\|\boldsymbol{V}_1\|_0^2 \le 4(1+\tilde{C})\big(\max_{\alpha,\beta=1,2}\|\overset{\times}{q}{}^{\alpha\beta}\|^2_{C^0([0,1]\times S^1)}\big)\big(\|\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_1 + \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_2\|_0^2 + \|\overset{\times}{D}{}'_1\underset{+}{\overset{\times}{v}}{}_2 - \overset{\times}{D}{}'_2\underset{+}{\overset{\times}{v}}{}_1\|_0^2\big). \tag{A.160}$$

From (A.152), (A.155), (A.158) and (A.160), we conclude that

$$\sqrt{\overset{\times}{g}}_{\min}\|\overset{\times}{\nabla}{}_h^{\perp\prime}\cdot\boldsymbol{u}\|^2_{\bar{\Omega}_{\rm g}} \le 2\big(\|\boldsymbol{V}_0\|_0^2 + \|\boldsymbol{V}_1\|_0^2\big) \le C_0\big(\|\boldsymbol{v}\|^2_{\underset{+}{\Omega}_{\rm c}} + \|\sqrt{\underset{+}{g}}^{\frac{1}{2}}\underset{+}{\overset{\times}{\nabla}}{}'_h\cdot\boldsymbol{v}\|^2_{\underset{+}{\bar{\Omega}}_{\rm g}} + \|\sqrt{\underset{+}{g}}^{\frac{1}{2}}\underset{+}{\overset{\times}{\nabla}}{}_h^{\perp\prime}\cdot\boldsymbol{v}\|^2_{\underset{+}{\bar{\Omega}}_{\rm g}}\big) \tag{A.161}$$

where C_0 is a positive constant depending only on $\|\overset{\times}{q}{}^{\alpha\beta}\|_{C^{0,1}([0,1]\times S^1)}$.

In view of (A.143), (A.151) and (A.161), the estimate (A.144) follows. ∎

We are now ready to complete the proof of the LBB estimate (3.57). Since $I^{-1}(q)-p\in\operatorname{span}\{\mathbf{1}_{\bar{\Omega}_{\rm go}},\mathbf{1}_{\bar{\Omega}_{\rm go}}\}$, from (2.29) and (2.26), we have

$$\langle\,\overset{\times}{\nabla}{}'_h\cdot\boldsymbol{u}\,,\,I^{-1}(q)-p\,\rangle_{\bar{\Omega}_{\rm g}} = -\langle\,\boldsymbol{u}\,,\,\overset{\times}{\nabla}_h(I^{-1}(q)-p)\,\rangle_{\Omega_{\rm c}} = 0. \tag{A.162}$$

From (A.143), (A.162) and (A.2),

$$\langle \overset{\times}{\nabla}'_h \cdot \boldsymbol{u}\, , \, p \, \rangle_{\bar{\Omega}_{\mathrm{g}}} = \langle \overset{\times}{\nabla}'_h \cdot \boldsymbol{u}\, , \, I^{-1}(q) \, \rangle_{\bar{\Omega}_{\mathrm{g}}} = \langle \underset{+}{\overset{\times}{\nabla}}'_h \cdot \boldsymbol{v}\, , \, q \, \rangle_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}} = \|q\|^2_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}}. \tag{A.163}$$

In view of (A.163), (A.140), (A.144) and (A.3), we conclude that

$$\frac{\langle \overset{\times}{\nabla}'_h \cdot \boldsymbol{u}\, , \, p \, \rangle_{\bar{\Omega}_{\mathrm{g}}}}{\|p\|_{\bar{\Omega}_{\mathrm{g}}} \|\boldsymbol{u}\|_{\overset{\times}{H}{}^1_h}} = \frac{\|q\|^2_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}}}{\|p\|_{\bar{\Omega}_{\mathrm{g}}} \|\boldsymbol{u}\|_{\overset{\times}{H}{}^1_h}} \geq \left(\frac{\sqrt{\underset{+}{g}}_{\mathrm{min}}}{\sqrt{g}_{\mathrm{max}}} \right)^{\frac{1}{2}} \frac{\|q\|_{\underset{+}{\bar{\Omega}}_{\mathrm{g}}}}{C^{\dagger} \|\boldsymbol{v}\|_{\underset{+}{\overset{\times}{H}}{}^1_h}} \geq \left(\frac{\sqrt{\underset{+}{g}}_{\mathrm{min}}}{\sqrt{g}_{\mathrm{max}}} \right)^{\frac{1}{2}} \frac{\gamma}{C^{\dagger}}. \tag{A.164}$$

Therefore (3.57) follows with $\beta = \left(\frac{\sqrt{\underset{+}{g}}_{\mathrm{min}}}{\sqrt{g}_{\mathrm{max}}} \right)^{\frac{1}{2}} \frac{\gamma}{C^{\dagger}} > 0$, which is independent of h.