CHAPTER 9
CONVERGENCEIN $\mathbf{R}^n$

9.1 LIMITS OF SEQUENCES

DEFINITION. Let $\{\mathbf{x}_n\}$ be a sequence points in $\mathbf{R}^n$.

(1) **$\{\mathbf{x}_n\}$ is said to converge to some point $\mathbf{a} \in \mathbf{R}^n$ (called the limit of $\mathbf{x}_n$) if and only if for every $\epsilon > 0$ there is $N \in \mathbf{N}$ such that**
$$k \geq N \quad \textbf{implies} \quad \|\mathbf{x_k} - \mathbf{a}\| < \epsilon.$$
(2) **$\{\mathbf{x}_n\}$ is said to be bounded if and only if there is $M > 0$ such that $\|\mathbf{x}_n\| \leq M$ for all $n \in \mathbf{N}$.**
(3) **$\{\mathbf{x}_n\}$ is said to be Cauchy if and only if for every $\epsilon > 0$ there is $N \in \mathbf{N}$ such that**
$$m, k \geq N \quad \textbf{implies} \quad \|\mathbf{x_m} - \mathbf{x_k}\| < \epsilon.$$

Theorem. *Let $\mathbf{a} = (a(1), \cdots, a(n))$ and $\mathbf{x}_k = (x_k(1), \cdots, x_k(n))$ belong to $\mathbf{R}^n$ for $k \in \mathbf{N}$. Then $\mathbf{x_k} \to \mathbf{a}$ as $k \to \infty$, if and only if the component sequences $x_k(j) \to a(j)$ as $k \to \infty$, for all $j = 1, 2, \cdots, n$.*

Theorem. *For each $\mathbf{a} \in \mathbf{R}^n$ there is a sequence $\mathbf{x}_k \in \mathbf{Q}^n$ such that $\mathbf{x_k} \to \mathbf{a}$ as $k \to \infty$.*

DEFINITION. A set E is said to be separable if and only if there is an at most countable subset $Z \subseteq E$ such that to each $\mathbf{a} \in E$ there is a sequence $\mathbf{x}_k \in Z$ such that $\mathbf{x_k} \to \mathbf{a}$ as $k \to \infty$.

Theorem. *Let $n \in \mathbf{R}$.*

(1) *A sequence in $\mathbf{R}^n$ can have at most one limit.*
(2) *If $\{\mathbf{x}_k\}$ is a sequence in $\mathbf{R}^n$ that converges to $\mathbf{a}$ and $\{\mathbf{x}_{k_j}\}_{j \in \mathbf{N}}$ is any subsequence of $\{\mathbf{x}_k\}$, then $\mathbf{x}_{k_j}$ converges to $\mathbf{a}$ as $j \to \infty$.*
(3) *Every convergent sequence in $\mathbf{R}^n$ is bounded ,but not conversely.*
(4) *Every convergent sequence in $\mathbf{R}^n$ is Cauchy.*
(5) *If $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ are convergent sequences in $\mathbf{R}^n$ and $\alpha \in \mathbf{R}$, then*
$$\lim_{k \to \infty} (\mathbf{x}_k + \mathbf{y}_k) = \lim_{k \to \infty} \mathbf{x}_k + \lim_{k \to \infty} \mathbf{y}_k,$$
$$\lim_{k \to \infty} (\alpha \mathbf{x}_k) = \alpha \lim_{k \to \infty} \mathbf{x}_k,$$
and
$$\lim_{k \to \infty})\mathbf{x}_k \cdot \mathbf{y}_k) = (\lim_{k \to \infty} \mathbf{x}_k) \cdot (\lim_{k \to \infty} \mathbf{y}_k).$$
Moreover, when $n = 3$,
$$\lim_{k \to \infty} (\mathbf{x}_k \times \mathbf{y}_k) = (\lim_{k \to \infty} \mathbf{x}_k) \times (\lim_{k \to \infty} \mathbf{y}_k).$$

Corollary. $\lim_{k\to\infty} \|\mathbf{x}_k\| = \|\lim_{k\to\infty} \mathbf{x}_k\|$.

Theorem. *[***BOLZANO-WEIERSTRASS THEOREM***]***. Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.**

Theorem. *A sequence $\{\mathbf{x}_k\}$ in $\mathbf{R}^n$ converges if and only if it is Cauchy.*

Theorem. *A sequence $\{\mathbf{x}_k\}$ in $\mathbf{R}^n$ converges to $\mathbf{a}$ if and only if for every open set V that contains $\mathbf{a}$, there is an $N \in \mathbf{N}$ such that $k \geq N$ implies $\mathbf{x}_k \in V$.*

Theorem. *Let $E \subseteq \mathbf{R}^n$. Then E is closed if and only if E contains all its limit points. i.e. if $\{\mathbf{x}_i\} \subset E$ and $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$ implies $\mathbf{x} \in E$.*

DEFINITION. Let E be a subset of $\mathbf{R}^n$.

(1) **An open covering of E is a collection of sets $\{V_a\}_{a\in A}$ such that each V_a is open and**

$$E \subseteq \cup_{a\in A} V_a.$$

(2) **The set is said to be compact if and only if every open covering of E has a finite subcovering ; i.e. if $\{V_a\}_{a\in A}$ is an open covering of E, then there is a finite subset A_0 of A such that $4E \subseteq \cup_{a\in A_0} V_a$.**

.

Lemma. *[***Borel Covering lemma***]***. Let E be closed bounded subset of $\mathbf{R}^n$. If $r : E \to (0,\infty)$, then there exists finitely many points $\mathbf{y}_1, \cdots, \mathbf{y}_N \in E$ such that**

$$E \subseteq \cup_{j=1}^{N} B_{r(\mathbf{y}_j)}(\mathbf{y}_j).$$

Theorem. *[***Heine-Borel Theorem***]***. Let E be a subset of $\mathbf{R}^n$. Then E is compact if and only if E is closed and bounded.**

Remark. $(0,1) = \cup_{n=1}^{\infty}(\frac{1}{n}, 1-\frac{1}{n})$, $[1,\infty) \subset \cup_{n=1}^{\infty}(1-\frac{1}{n}, n)$.

9.2 LIMITS OF FUNCTIONS $f : \mathbf{A} \subset \mathbf{R}^n \to \mathbf{R}^m$.

Example. (a) $f(x,y) = (\log(xy - y + 2x - 2), \sqrt{9 - x^2 - y^2})$
(b) $g(x,y) = (\sqrt{1-x^2}, \log(x^2 - y^2), \sin x \cos y)$.

DEFINITION. Let $m, n \in \mathbf{N}$ and $\mathbf{a} \in \mathbf{R}^n$, let V be an open set which contains a, and suppose that $f : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$. Then $f(\mathbf{x})$ is said to converge to $\mathbf{L}$ as x approaches a , if and only if for every $\epsilon > 0$ there is a $\delta > 0$(which in general depends on $\epsilon, f, V$and a) such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \text{ implies } \|f(\mathbf{x}) - \mathbf{L}\| < \epsilon.$$

In this case we write

$$\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$$

and call L the limit of $f(\mathbf{x})$ as x approaches a.

Theorem. *Let $m, n \in \mathbf{N}$ and $\mathbf{a} \in \mathbf{R}^n$, let V be an open set which contains $\mathbf{a}$, and suppose that $f, g : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$.*

(1) *If $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in V \setminus \{\mathbf{a}\}$ and $f(\mathbf{x})$ has a limit as $\mathbf{x} \to \mathbf{a}$, then $g(\mathbf{x})$ also has a limit as $\mathbf{x} \to \mathbf{a}$ and*

$$\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$

(2) *$\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists if and only if $f(\mathbf{x}_k) \to \mathbf{L}$ as $k \to \infty$ for every sequence $\mathbf{x}_k \in V \setminus \{\mathbf{a}\}$ which converges to $\mathbf{a}$ as $k \to \infty$.*

(3) *Suppose that $\alpha \in \mathbf{R}$. If $f(\mathbf{x})$ and $g(\mathbf{x})$ have limits as $\mathbf{x} \to \mathbf{a}$, then so do $(f + g)(\mathbf{x}), (\alpha f)(\mathbf{x}), f \cdot g(\mathbf{x})$ and $\|f(\mathbf{x})\|$. In fact*

$$\lim_{\mathbf{x}\to\mathbf{a}} (f+g)(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (\alpha f)(\mathbf{x}) = \alpha(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})),$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (f\cdot g)(\mathbf{x}) = (\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})) \cdot (\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})),$$

and

$$\|\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\| = \lim_{\mathbf{x}\to\mathbf{a}} \|f(\mathbf{x})\|.$$

Moreover when $m = 3$,

$$\lim_{\mathbf{x}\to\mathbf{a}} (f\times g)(\mathbf{x}) = (\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})) \times (\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})),$$

and when $m = 1$ and the limit of g is nonzero,

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/g(\mathbf{x}) = (\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}))/(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})).$$

(4) *Suppose that $f, g, h : V \setminus \{\mathbf{a}\} \to \mathbf{R}$ and $g(\mathbf{x}) \le h(\mathbf{x}) \le f(\mathbf{x})$ for all $\mathbf{x} \in V \setminus \{\mathbf{a}\}$. If*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}),$$

then the limit of h as $\mathbf{x} \to \mathbf{a}$ also exists and

$$\lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = \mathbf{L}.$$

(5) *Suppose that U is open in $\mathbf{R}^m$, that $\mathbf{L} \in U$, and $h : U \setminus \{\mathbf{L}\} \to \mathbf{R}^p$ for some $p \in \mathbf{N}$. If $\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ and $\mathbf{M} = \lim_{\mathbf{y}\to\mathbf{L}} h(\mathbf{y})$, then*

$$\lim_{\mathbf{x}\to\mathbf{a}} h \circ g(\mathbf{x}) = h(\mathbf{L}).$$

Theorem. *Let* $\mathbf{a} \in \mathbf{R}^n$, *let* V *be an open ball that contains* $\mathbf{a}$, *let* $f = (f_1, \cdots, f_m) : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$, *and let* $\mathbf{L} = (L_1, \cdots, L_m) \in \mathbf{R}^m$. *Then*

$$\mathbf{L} = \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}),$$

if and only if

$$L_j = \lim_{\mathbf{x} \to \mathbf{a}} f_j(\mathbf{x})$$

exist for each $j = 1, \cdots, m$).

Example. (a) Find $\lim_{(x,y)\to(0,0)}(3xy + 1, e^y + 2)$,
(b) $\lim_{(x,y)\to(0,0)} = \frac{2+x-y}{1+2x^2+3y^2}$.

Example. $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$.

Example. $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

Example. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

Example. Find iterate limits of $\frac{x^2}{x^2+y^2}$ at $(0, 0)$.

Remark. Suppose that I, J are open intervals, that $a \in I$ and $b \in J$ and $f : I \times J \setminus \{(a, b)\} \to \mathbf{R}$. If $g(x) = \lim_{y\to b} f(x, y)$ exists for each $x \in I \setminus \{a\}$, and if $h(y) = \lim_{x\to a} f(x, y)$ exists for each $y \in J \setminus \{b\}$, and if $f(x, y) \to L$ as $(x, y) \to (a, b)$ in $\mathbf{R}^2$. Then

$$L = \lim_{x\to a} \lim_{y\to b} f(x, y) = \lim_{y\to b} \lim_{x\to a} f(x, y).$$

9.3 CONTINUOUS FUNCTIONS

DEFINITION. Let E be a nonempty subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$.

(1) **f is said to be continuous at $\mathbf{a} \in E$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on $\epsilon, f, \mathbf{a}$) such that**

$$\|\mathbf{x} - \mathbf{a}\| < \delta \textbf{ and } \mathbf{x} \in E \textbf{ imply } \|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon.$$

(2) **f is said to be continuous on E (notation :$f : E \to \mathbf{R}^m$ is continuous) if and only if f is continuous at each $\mathbf{x} \in E$.**

DEFINITION. Let E be a nonempty subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$. Then f is said to be uniformly continuous on E (notation $f : E \to \mathbf{R}^m$ is unformly continuous) if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{a}\| < \delta \textbf{ and } \mathbf{x}, \mathbf{a} \in E \textbf{ imply } \|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon.$$

Theorem. *Let E be a nonempty compact subset of $\mathbf{R}^n$. If f is continuous on E, then f is uniformly continuous on E.*

Theorem. *Let $m, n \in \mathbf{N}$ and $f : \mathbf{R}^n \to \mathbf{R}^m$. Thenthe following three conditions are equivalent.*

(1) *f is continuous on $\mathbf{R}^n$.*
(2) *$f^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.*
(3) *$f^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.*

Theorem. *Let $m, n \in \mathbf{N}$, let E be a nonempty open subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$. Then f is continuous on E if and only if $f^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.*

Example. (a) $f(x) = \frac{1}{x^2+1}, E = (0, 1]$, (b) $f(x) = x^2, E = (1, 4)$.

Example. (a) $f(x) = x^2, V = (-1, 1)$, (b) $f(x) = 1/x, E = [1, \infty)$.

Theorem. *Let $m, n \in \mathbf{N}$. If H is a compact subset in $\mathbf{R}^n$ and $f : H \to \mathbf{R}^m$ is continuous on H, then $f(H)$ is a compact subset of $\mathbf{R}^m$.*

Theorem. *Let $m, n \in \mathbf{N}$. If E is a connected subset in $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$ is continuous on E, then $f(E)$ is a connected subset of $\mathbf{R}^m$.*

Remark. The graph of a continuous real function f on an interval $[a, b]$ is compact and connected.

Theorem. *[***EXTREME VALUE THEOREM***]***. Suppose that H is a nonempty compact subset of $\mathbf{R}^n$ and $f : H \to \mathbf{R}$ is continuous, then**

$$M = \sup\{f(\mathbf{x}) : \mathbf{x} \in H\} \textbf{ and } m = \inf\{f(\mathbf{x}) : \mathbf{x} \in H\}$$

are finite real numbers. Moreover there exist points $\mathbf{x}_M, \mathbf{x}_m \in H$ such that $M = f(\mathbf{x}_M), m = f(\mathbf{x}_m)$.

Theorem. *Let $m, n \in \mathbf{N}$. If H is a compact subset in $\mathbf{R}^n$ and $f : H \to \mathbf{R}^m$ is 1-1 and continuous on H, then f^{-1} is continuous on $f(H)$.*

Remark. If $a_j \le b_j$ for $j = i, \cdots, n$, then

$$R = \{(x_1, \cdots, x_n) : a_j \le x_j \le b_j\}$$

is conncted.

9.4 COMPACT SETS

Remark. All finite sets are compact.

Remark. A closed subset of a compact set is compact.

Theorem. *[***LINDERLÖF***]***. Let $n \in \mathbf{N}$ and E be a subset of $\mathbf{R}^n$. If $\{V_a\}_{a\in A}$ is a familly of open subsets such that $E \subset \cup_{a\in A} V_a$, then there is an at most countable subset A_0 of A such that $E \subset \cup_{a\in A_0} V_a$.**

9.5 APPLICATIONS

Theorem. *[DINI]. Suppose that H is a compact subset in $\mathbf{R}^n$ and $f_k : H \to \mathbf{R}$ is a sequence of pointwise monotone sequence of continuous functions. If $f_k \to f$ pointwise on H as $k \to \infty$ and f is continuous on H, then $f_k \to f$ uniformly on H. In particular if ϕ_k is a pointwise monotone sequence of continuous functions on $[a, b]$ that converges pointwise to a continuous function, then*

$$\lim_{k\to\infty} \int_a^b \phi_k(x)dx = \int_a^b \lim_{k\to\infty} \phi_k(x)dx.$$

DEFINITION.

(1) **A set $E \subset \mathbf{R}$ is said of measure zero if and only if for every $\epsilon > 0$ there is a countable collection of intervals $\{I_k\}_{k\in\mathbf{N}}$ that covers E such that $\sum_{k=1}^{\infty} |I_k| < \epsilon$.**

(2) **A function $f : [a, b] \to \mathbf{R}$ is said to be almost everywhere continuous on $[a, b]$ if and only if the set of points $x \in [a, b]$ where f is discontinuous is a set of measure zero.**

Remark. Every at most countable set of real numbers is a sets of measure zero.

Remark. Let $\{E_k\}_{k\in\mathbf{N}}$ be a sequence of set of measure zero, then $\cup_k \in \mathbf{N}$ is a set of measure zero.

DEFINITION. Let $f : [a, b] \to \mathbf{R}$ be bounded.

(1) **The oscillation of f on an interval J that intersects with $[a, b]$ is defined to be**

$$\Omega_f(J) = \sup_{x,y\in J\cap[a,b]} |f(x) - f(y)|.$$

(2) **The oscillation of f at a point $t \in [a, b]$ is defined to be**

$$\omega_f(t) = \lim_{h\to 0^+} \Omega_f((t-h, t+h))$$

when the limit exists.

Remark. If $f : [a, b] \to \mathbf{R}$ is bounded, then $\omega_f(t)$ exists for all $t \in [a, b]$ and is finite.

Remark. Let $f : [a, b] \to \mathbf{R}$ be bounded. If E represents the set of point of discontinuity of f on $[a, b]$, then

$$E = \cup_{j=1}^{\infty} \{t \in [a, b] : \omega_f(t) \geq \frac{1}{j}\}.$$

Lemma. *Let* $f : [a, b] \to \mathbf{R}$ *be bounded. For each* $\epsilon > 0$*, the set* $H = \{t \in [a, b] : \omega_f(t) \geq \epsilon\}$ *is compact.*

Lemma. *Let* I *be a bounded closed interval and* $f : I \to \mathbf{R}$ *be bounded. If* $\epsilon > 0$ *and* $\omega_f(t) < \epsilon$ *for all* $t \in I$*, then there is a* $\delta > 0$ *such that* $\Omega_f(J) < \epsilon$ *for all closed intervals* $J \subseteq I$ *that satisfy* $|J| < \delta$*.*

Theorem. *[LEBESGUE]. Let* $f : [a, b] \to \mathbf{R}$ *be bounded. Then* f *is Riemann integrable on* $[a, b]$ *if and only if* f *is almost everywhere continuous on* $[a, b]$*. In particular if* f *has at most countable many discontinuity points on* $[a, b]$*, then* f *is integrable on* $[a, b]$*.*

Corollary. *If* $f : [a, b] \to [0, \infty)$ *is Riemann integrable, then so is* f^α *for every* $\alpha > 0$*.*

Theorem. *[CLOSED GRAPH THEOREM]. Let* I *be a closed interval and* $f : I \to \mathbf{R}$*. Then* f *is continuous on* I *if and only if the graph of* f *is closed and connected in* $\mathbf{R}^2$*.*