ADVANCED CALCULUS (JUNE 17, 2008)

1. (a) $\int_0^1 \int_y^1 \sin(x^2) dxdy = \int_0^1 \int_0^x \sin(x^2) dydx = \int_0^1 x\sin(x^2)dx = (1-\cos 1)/2.$
(b) $\int_0^1 \int_{z^2}^{\sqrt{z}} x^3 dxdz = \int_0^1 (z^2 - z^8)/4 dz == (1/3 - 1/9)/4 = 1/18.$

2. (a) $\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial g}{\partial y}\cos\theta - \frac{\partial g}{\partial x}\sin\theta,$
$\frac{\partial v}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial g}{\partial x}(-r\sin\theta) + \frac{\partial g}{\partial y}(r\cos\theta).$
(b) $f_x = 1, g_x = 2x, g_y = 1$ so $F_x(f,g) = g_x(f,g)f_x + g_y(f,g)g_x = 2(x+e^y)*1 + 1*2x.$

3. Since $|f(x,y)| \le r|\cos^3\theta + \sin^\theta| \le 2r$, so f is continuous at $(0,0)$.
By $\lim_{x\to 0}\frac{f(x,0)}{x} = 1$ so $f_x(0,0) = 1$ and also $f_y(0,0) = 1$.
Since $|[f(x,y) - f(0,0) - D^1 f((0,0),(x,y))]|/\|(x,y)\| = |x^2y + xy^2|/r^3$,
which does not go to 0 as $(x,y) \to (0.0)$.

4. Since $\nabla f = (1,1,3)$ there is no interior extremum. On the bottom part of the boundary , by Lagrange $(1,1,3) = a(2x,2y,-1)$, so $a = -3, x = y = -1/6$, then $f(-1/6,-1/6,1/18) = -1/6$. On the top part of the boundary $(1,1,3) \neq (0,0,1)$. On the top edge, $(1,1,3) = b(2x,2y,-1) + c(0,0,1)$, we have $z = 4, x = y = \pm\sqrt{2}$. Then $f(-\sqrt{z}, -\sqrt{2}, 4) = -2\sqrt{2} + 12$ and $f(\sqrt{z}, \sqrt{2}, 4) = 2\sqrt{2} + 12$.

5. y $mt^{m-1}f(\mathbf{x}) = \frac{d}{dt}t^m f(\mathbf{x}) = \frac{d}{dt}f(t\mathbf{x}) = \sum_1^n x_j f_{x_j}(t\mathbf{x})$. Then let $t = 1$.

6. (a) $\nabla f = (0,0)$ implies $y = k\pi$ and $x = (l+1/2)\pi$. $f(l+1/2)\pi, k\pi) = 2$ if both k,l are even, $f(l+1/2)\pi, k\pi) = -2$ if both k,l are odd, and $f(l+1/2)\pi, k\pi) = 0$ if $k+l$ are odd.
(b) $\nabla f = \mathbf{0}$ implies $2x + 2y = 0, 2x = 6y = 0$, so $(x,y) = (0,0)$ which is not in the triangle. On the boundary edge $L[(1,0),(3,2)], y = x-1, f = 6x^2 - 8x + 3$, so f is increasing there. On the boundary edge $L[(1,0),(2,-1)], y = -x+1, f = 2x^2 - 4x + 3$, so f is increasing there. On the boundary edge $L[(2,-1),(3,2)], y = 3x-7, f = 34x^2 - 140x + 147$. The minimum is at $x = 140/68$ there. So $f(1,1)$ is minimum and $f(3,2)$ is maximum.

7. Let $u = x+y, v = xy$, solve x,y in terms of u,v, you get $x = u/2 + \sqrt{(u/2)^2 - v}, y = u/2 - \sqrt{(u/2)^2 - v}$, that gives the inverse of f from F to E.

8. Let $y - x = u, y + x = v$, then $\iint_E e^{(y-x)/(y+x)} dA = 1/2 \int_2^4 \int_{-v}^0 e^{u/v} dudv = 3(1 - e^{-1})$.