CHAPTER 2, PRELUDE TO CALCULUS

2.1 Tangent Lines and Slope Predictors

Slope predictor.

Example.

(1) The tangent line of $f(x) = x^2$ at (a, a^2) $M_a = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = 2a$

Slope predictor. *For general $f(x)$ the slope predictor of the tangent at $(a, f(a))$ is*

$$M_a = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Theorem. *If $f(x) = px^2 + qx + r$ then $M_a = 2pa + q$*

Normal line : $y - f(a) = -\frac{1}{M_a}(x - a)$

2.2 The Limit Concept

The Limit. *Suppose that $f(x)$ is defined in an open interval containing the point a (except possibly not at a itself). Then we say that the number L is the limit of $f(x)$ as x approaches a and we write $\lim_{x\to a} f(x) = L$ provided that the following criterion is satisfied: given any $\epsilon >$ there exists a corresponding δ such that $|f(x) - L| < \epsilon$ for all x such that $0 < |x - a| < \delta$.*

i.e.

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\forall\epsilon > 0, \exists\delta_f(\epsilon) > 0 \text{ such that } \quad 0 < |x - a| < \delta_f(\epsilon) \implies |f(x) - L| < \epsilon \tag{1}$$

.

($\forall\epsilon$ is for every ϵ, and $\exists\delta$ is there exists δ.)

Remark(1). *If (1) holds for ϵ and $\epsilon < \epsilon'$, then (1) holds for ϵ' too. So we may just concentrate on $\epsilon \le a$ or $\epsilon = \frac{1}{n}$ or $\epsilon = \frac{1}{2^n}$.*

Remark(2). *If (1) holds for a fixed ϵ, then (1) holds for any $\delta' < \delta$, so we may only look for $\delta < b$.*

Remark(3). *If (1) is not true, then for some ϵ_0, for any n there is x_n such that $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - L| \ge \epsilon_0$. So we have a sequence $\{x_n\}$ goes to a but $f(x_n)$ stay away from L with distance at least ϵ_0.*

Remark(4). *The limit if exists, is unique.*

Proof. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$ and $L \neq M$, then $|L - M| > 0$. Let $\epsilon = \frac{1}{3}|L - M|$, for x near a we have $|f(x) - L| < \epsilon,\ |f(x) - M| < \epsilon$. But then

$$|L - M| = |L - f(x) - M + f(x)| \leq |f(x) - L| + |f(x) - M| < 2\epsilon < |L - M|$$

which is a contradiction.

Remark(5). *If $\lim_{x\to a} f(x) = L$, then $f(x)$ is bounded on some interval contains a.*

Remark(6). *If $|f(x) - L| \leq K|x - a|$, then $\delta = \frac{\epsilon}{K}$.*

Rematk(7). *If $f(x) = g(x)$ for all $x, 0 < |x - a| < \lambda$, then $\lim_{x\to a} f(x) = L$ if and onlu if $\lim_{x\to a} g(x) = L$*

Rematk(8). *If $f(x) \leq K$ and $\lim_{x\to a} f(x) = L$, then $L \leq K$.*

Example.

(1) $\lim_{x\to 3} \frac{x-1}{x+2} = \frac{2}{5}$,
(2) $\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \frac{4}{5}$,
(3) $\lim_{t\to 0} \frac{\sqrt{t+25}-5}{t} = \frac{1}{10}$,
(4) $\lim_{x\to 0} \frac{x}{|x|}$ does not exists,
(5) $F(x) = 1$ if $x \neq 0$ and $= 0$ if $x = 0$, $\lim_{x\to 0} F(x) = 1$

The Limit Laws. *If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then*

(1) $\lim_{x\to a} cf(x) = cL$,
(2) $\lim_{x\to a}(f + g)(x) = L + M$,
(3) $\lim_{x\to a}(f - g)(x) = L - M$,
(4) $\lim_{x\to a}(f \cdot g)(x) = L \cdot M$
(5) $\lim_{x\to a}(f/g)(x) = L/M$ *if* $M \neq 0$,
(6) *Suppose that $a > 0$ then $\lim_{x\to a} x^{1/n} = a^{1/n}$ (root law),*
(7) *Suppose that $\lim_{x\to a} g(x) = M$ and $\lim_{y\to M} f(y) = f(M)$, then $\lim_{x\to a} f \circ g(x) = f(M)$ (substituition law),*
(8) *Suppose that $f(x) \geq g(x) \geq h(x)$ and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$ (squeeze law).*

Proof of (1). Since $|cf(x) - cL| = |c||f(x) - L|$, let $\delta = \delta_f(\frac{\epsilon}{|c|})$.

Proof of (2). Since $|f(x)+g(x)-L-M| \leq |f(x)-L|+|g(x)-M|$, let $\delta = \min(\delta_f(\frac{\epsilon}{2}), \delta_g(\frac{\epsilon}{2}))$.

Proof of (4). Since $|fg(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$. Choose δ_1 such that for $0 < |x - a| < \delta_1$ we have $|f(x) - L| < \frac{\epsilon}{2|M|}$. Now choose δ_2 such that for $0 < |x - a|, \delta$ we have $|g(x) - M| < \frac{\epsilon}{2(|L|+\delta_1)}$ Let $\delta = \min(\delta_1, \delta_2)$, then for $0 < |x - a| < \delta$ we have $|fg(x) - LM| < \epsilon$.

Proof of (5). Assume that $f(x) = 1$, then $|\frac{1}{g(x)} - \frac{1}{M}| = \frac{|g(x)-M|}{|g(x)||M|}$, let $\delta = \min(\delta_g(1), \delta_g(\frac{\epsilon}{|M|(|M|+1)}))$. Combine with (4) we can prove the general case.

Proof of (7). Given $\epsilon > 0$, since $\lim_{y\to M} f(y) = f(M)$, there is $\delta_1 > 0$ such that $|f(y) - f(M)| < \epsilon$ when $0 < |y - M| < \delta_1$. Also since $\lim_{x\to a} g(x) = M$, there exists $\delta > 0$ such that $|g(x) - M| < \delta_1$ when $0 < |x - a| < \epsilon$. Put them together, we have that for if $0 < |x - a| < \delta$,then $|f \circ g(x) - f(M)| < \epsilon$.

Proof of (8). Let $\delta = \min(\delta_f(\epsilon), \delta_h(\epsilon))$, then for $0 < |x - a| < \delta$, we have

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

.

Example.

(1) $\lim_{x\to 3} x^2 + 2x + 4 = 19$,
(2) $\lim_{x\to 3} \frac{2x+5}{x^2+2x+4} = \frac{11}{19}$,
(3) $\lim_{x\to 1} \frac{1}{(x-1)^2}$ does not exist,
(4) $\lim_{x\to 2} \frac{x^2-4}{x^2+x-6}$,
(5) $\lim_{x\to 4}(3\sqrt{x^3} + 20\sqrt{x})^{1/3} = 4$,
(6) $px^2 + qx + r$(slpoe predictor) ,
(7) $\frac{1}{\sqrt{x}}$(slope predictor),
(8) $x + \frac{1}{x}$ (slope predictor),
(9) $\lim_{x\to 0} x \sin \frac{1}{x}$,
(10) Let $F(x) = 1$ if $x \neq 0$ and $F(0) = 0$, $g(x) = x \sin \frac{1}{x}$, then $\lim_{x\to 0} F \circ g(x)$ does not exist,
(11) $\lim_{x\to 0} \sin x = 0$,
(12) $\lim_{x\to 0} \cos x = 1$.

2.3 More about Limits

Theorem. $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example.

(1) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$,
(2) $\lim_{x\to 0} \frac{\tan 3x}{x} = 3$,
(3) $\lim_{x\to 0} \sin \frac{\pi}{x}$ does not exist,
(4) $g(x) = x \sin \frac{1}{x}$ for $x \neq 0$, then $\lim_{x\to 0} g(x) = 0$.

One Side Limit.

Right side limit. $\lim_{x\to a^+} f(x) = L$ *iff for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $0 < x - a < \delta$ *implies that* $|f(x) - L| < \epsilon$.

Left side limit. $\lim_{x\to a^-} f(x) = L$ *iff for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $0 < a - x < \delta$ *implies that* $|f(x) - L| < \epsilon$.

Theorem. $\lim_{x\to a} f(x) = L$ *iff* $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

Example.

(1) $[[x]]$,
(2) $\sqrt{x}$,
(3) $f(x) = x^2$ if $x \le 0$ and $f(x) = x \sin \frac{1}{x}$ if $x > 0$,
(4) $\lim_{x\to 3^-} \frac{x^2}{x^2+1} + \sqrt{9 - x^2}$,
(5) Exists of tangent line at $a = 0$ for $|x|$.

Limit ∞.

Limit ∞. $\lim_{x\to a} f(x) = \infty$ *iff for any* $K > 0$ *there exists* $\delta > 0$ *such that* $0 < |x - a| < \delta$ *implies that* $f(x) > K$.

Rematk. *If* $\lim_{x\to a^\pm} f(x) = \pm\infty$, *then* $x = a$ *is an asymptote of the graph of* f.

Example.

(1) $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$,
(2) $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$,
(3) $\lim_{x\to 1^+} \frac{2x+1}{x-1} = \infty$ and $\lim_{x\to 1^-} \frac{2x+1}{x-1} = -\infty$,
(4) $\lim_{x\to 0} \frac{1}{x} \sin \frac{1}{x}$ does not exist.

2.4 The Concept of Continuity

Continuous at a point. *Suppose that* $f(x)$ *is defined in am open interval contains* a, *then* $f(x)$ *is contiuous at* a *iff* $\lim_{x\to a} f(x) = f(a)$.

If f *is continuous at every point in the doamin, we say that* f *is a continuous function.*

Example.

(1) $\frac{1}{x-2}$ continuous at $x \neq 2$,
(2) $g(x) = 1$ if $x \ge 0$ and $g(x) = -1$ if $x < 0$, then g has a jump discontinuity at 0,
(3) $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 0$, then $f(x)$ has a removable discontinuity at 0.
(4) $x - [[x]]$ has jump discontinuity at every integer point.

Combinations of Continuous Functions.

Theorem. *Suppose that* f, g *are continuous functions, then* $cf, f + g, f - g, f \cdot g$ *are continuous functions and* f/g *is continuous* x *whenever* $g(x) \neq 0$

Example.

(1) $x^3 - 3x^2 + 1$,
(2) $\frac{x-2}{x^2-3x+2}$ is continuous at all $x \neq 1, 2$.

Trigonometric Functions.

Theorem. $\cos x, \sin x$ *are continuous functions on* $\mathbb{R}$.

Corollary. $\tan x, \cot x, \sec x, \csc x$ *are continuous functions on its domain.*

Composition of Continuous Functions.

Tneorem. *If* g *is continuous at* a *and* f *is continuous at* $g(a)$*, then* $f \circ g$ *is continuous at* a*.*

Example. $(\frac{x-7}{x^2+2x+2})^{2/3}$.

Continuous Functions on Closed Interval. *A function* f *defined on* $[a, b]$ *if it is continuous on* (a, b) *and* $\lim_{x \to a^+} f(x) = f(a), \lim_{x \to b^-} f(x) = f(b)$*.*

Theorem. *Suppose that* f *is a continuous on* $[a, b]$*, then tere exist* x_1, x_2 *in* $[a, b]$ *such that* $f(x_1) \geq f(x)$ *and* $f(x_2) \leq f(x)$ *for all* $x \in [a, b]$*.*

Remark. *The theorem above means that continuous function on bounded closed interval takes both maximun and minmum. Ingeneral this not true for function continuous on open interval.*

Theorem. *Suppose that* f *is continuous at* a *and* $f(a) > 0$ *then there is an open interval* I *contains* a *such that* $f(x) > 0$ *for all* $x \in I$*.*

Corollary. *Suppose that* f *is continuous at* a *and* $f(a) > c, (f(a) < c)$ *then there is an open interval* I *contains* a *such that* $f(x) > c, (f(x) > c)$ *for all* $x \in I$*.*

Intermediate Value Theorem.

Theorem. *Suppose that* f *is continuous on* $[a, b]$ *and* $f(a) \neq f(b)$*. Then for any* k *between* $f(a), f(b)$*, there is a* $c \in [a.b]$ *such that* $f(c) = k$*.*

Proof. Let $S = \{x \in [a, b] : f(x) \leq k\}$, it is clear that $a \in S$ and b is an upper bound of S. Let c be the least upper bound of S. If $f(c) < k$ then there is an open interval I contains c such that $f(x) < k$ for all $x \in I$, that means there are $x > c$ such that $f(x) < K$, which is a contradiction to that c is the least bound of S. If $f(c) > k$, then for smoe $\delta > 0$ such that $f(x) > K$ for all $x \in (c - \delta, c + \delta)$. But then $c - \delta$ is not a upper bound of S, there will be a $x_1 \in S$ such that $c - \delta < x_1 \leq c$, that is a contradiction too. So $f(c) = k$

Example.

(1) $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x \geq 0$,
(2) $x^2 - 2$,
(3) $x^3 - x - 2$ has root in $(0, 2)$.
(4) $\frac{10[[1000x]]-4995}{10000}$,
(5) If f is a continuous function on $[0, 1]$ with value in $[0, 1]$, then there is $a \in [0, 1]$ such that $f(a) = a$.