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Existence and multiplicity of boundary blow-up nonnegative solutions to two-point boundary value problems☆

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1. Introduction

In this paper we investigate the necessary, sufficient conditions for the existence and the multiplicity of boundary blow-up nonnegative solutions of the two-point boundary value problem

$$-u''(x)=\lambda f(u(x)),\quad 0<x<1, \tag{1.1}$$

$$\lim_{x\to 0^+} u(x)=\infty=\lim_{x\to 1^-} u(x), \tag{1.2}$$

where λ is a positive bifurcation parameter and f is a Lipschitz continuous function.

Boundary blow-up solutions of the problem

$$-\Delta u(x)=f(u(x)),\quad x\in\Omega \tag{1.3}$$

$$u\,|_{\partial\Omega}=\infty, \tag{1.4}$$

where Ω is bounded domain in $\Re^N$ $(N\geq 1)$ have been extensively studied, see [1–3, 6–14]. A problem of this type was first considered by Bieberbach [2] in 1916,

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where $f(u) = -e^u$ and $N = 2$. Bieberbach showed that if Ω is a bounded domain in R^2 such that $\partial\Omega$ is a C^2 submanifold of R^2, then there exists a unique $u \in C^2(\Omega)$ such that $-\Delta u(x) = -e^u$ in Ω and $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. Here $d(x)$ denotes the distance from a point x to $\partial\Omega$. Rademacher [14], using the idea of Bieberbach, extended to smooth bounded domain in R^3. In this case the problem plays an important role, when $N = 2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, according to [14], in the study of the electric potential in a glowing hollow metal body. Lazer and McKenna [8] extended the results for Ω a bounded domain in R^N $(N \geq 1)$ satisfying a uniform external sphere condition and the nonlinearity $f = f(x,u) = p(x)e^u$, where $p(x)$ is continuous and strictly negative on $\bar{\Omega}$. Very recently, Lazer and McKenna [9] obtained similar results when Δ is replaced by the Monge–Ampére operator and Ω is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f = p(x)u^a$ with $a > 1$. Posteraro [13], for $f(u) = -e^u$ and $N \geq 2$, proved estimates for the solution $u(x)$ of (1.3)–(1.4) and for the measure of Ω comparing this problem with a problem of the same type defined in a ball. In particular, when $N = 2$, Posteraro obtained an explicit estimate of the minimum of $u(x)$ in terms of the measure of Ω:

$$\min_{x \in \Omega} u(x) \geq \ln(8\pi/|\Omega|).$$

The existence of solutions of Problem (1.3), (1.4), with f monotone, but not uniqueness, was studied by Keller [6]. For $f(u) = -u^a$ with $a > 1$, Problem (1.3), (1.4) is of interest in the study of the subsonic motion of a gas when $a = 2$ (see [12]) and is related to a problem involving superdiffusion, particularly for $1 < a \leq 2$ (see [4, 5]). Pohozaev [12] proved existence, but not uniqueness, for Problem (1.3), (1.4), when $f(u) = -u^2$. For the case where $f(u) = -u^{(N+2)/(N-2)}$ $(N > 2)$ Loewener and Nirenberg [10] proved that if $\partial\Omega$ consists of the disjoint union of finitely compact C^∞ manifolds, each having codimension less than $N/2 + 1$, then there exists a unique solution of Problem (1.3), (1.4). The uniqueness was established for $f(u) = -u^a$ with $a > 3$, when $\partial\Omega$ is a C^2-manifold and Δ is replaced by a more general second order elliptic operator, by Kondrat'ev and Nikishkin [7]. Marcus and Véron [11] proved uniqueness for $f(u) = -u^a$ with $a > 1$, when $\partial\Omega$ is compact and is locally the graph of a continuous function defined on an $(N-1)$-dimensional space. Recently, Díaz and Letelier [3] have proved existence and uniqueness when Δ is replaced by the p-Laplacian, $f(u) = -u^a$ with $a > 1$, and $\partial\Omega$ of class C^2.

For general nonlinearities $f(u)$ and in one space dimension, very recently, Anuradha et al. [1] considered Problem (1.1), (1.2). They mainly proved the existence and multiplicity of boundary blow-up nonnegative solutions basing on building a quadrature method for such boundary blow-up solutions as follows:

Define

$$F(s) := \int_0^s f(t)\,dt,$$

$$I := \{s \in R^+ \cup \{0\}\colon f(s) < 0 \text{ and } F(s) > F(u) \text{ for all } u > s\}. \tag{1.5}$$

Suppose that u is a nonnegative solution of Problem (1.1), (1.2). Let

$$\rho := \min_{x\in(0,1)} u(x). \tag{1.6}$$

Anuradha et al. [1] obtained the next lemma and theorems.

Lemma 1.1 (Anuradha et al. [1, Lemma 2.1]). *Given $\lambda > 0$ and f Lipschitz continuous, there exists a unique solution to Problem* (1.1), (1.2) *with* $\min_{x\in(0,1)} u(x) = \rho$ *if and only if*

$$G(\rho) := \sqrt{2}\int_{\rho}^{\infty} \frac{\mathrm{d}u}{\sqrt{F(\rho)-F(u)}} = \sqrt{\lambda} \quad \textit{for } \rho \in I. \tag{1.7}$$

Theorem 1.2 (Anuradha et al. [1, Theorem 3.1]). *If there exists any solution to Problem* (1.1), (1.2) *for any $\lambda > 0$, then*

$$\limsup_{u\to\infty} \frac{-f(u)}{u} = \infty. \tag{1.8}$$

Theorem 1.3 (Anuradha et al. [1, Theorem 3.2]). *Assume that there exists $a > 1$ such that*

$$\lim_{u\to\infty} \frac{-f(u)}{u^a} = \infty, \tag{1.9}$$

then there exist solutions to Problem (1.1), (1.2) *for some $\lambda > 0$. Furthermore, $G(\rho)$ is well defined and continuous for all $\rho \in I$.*

Remark 1. *Anuradha et al.* [1, Remarks 3.1 and 3.2] *showed that, in Problem* (1.1), (1.2),

(i) *for $f(u) = -u(\ln u + 1)$ satisfying Eq.* (1.8), *there are no solutions,*

(ii) *for $f(u) = -u(2(\ln u)^3 + 3(\ln u)^2)$ not satisfying Eq.* (1.9), *there exist solutions for some $\lambda > 0$.*

So

(iii) *condition* (1.8) *is necessary, but not sufficient for existence, and*

(iv) *condition* (1.9) *is sufficient, but not necessary for existence.*

Theorem 1.4 (Anuradha et al. [1, Theorem 3.3]). *If $f(u)$ satisfies Eq.* (1.9), *then*

$$G(\rho) \to 0^+ \quad \textit{as } \rho \to \infty.$$

Theorem 1.5 (Anuradha et al. [1, Theorem 3.7 and Remark 3.3], see also Fig. 7). *Let f satisfy Eq.* (1.9). *Assume that there exists $\beta > 0$ such that $f(\beta) = 0$ and $f(u) < 0$ for all $u > 0$ with $u \neq \beta$.*

(i) *If $f(0) < 0$, then there exists $\lambda^* > 0$ such that Problem* (1.1), (1.2) *has at least two nonnegative solutions for $\lambda > \lambda^*$, at least one nonnegative solution for $\lambda = \lambda^*$, and exactly one solution for $0 < \lambda < \lambda^*$.*

(ii) *If* $f(0)=0$, *then there exists* $\lambda^* > 0$ *such that Problem* (1.1), (1.2) *has at least three nonnegative solutions for* $\lambda > \lambda^*$, *at least two nonnegative solution for* $\lambda = \lambda^*$, *and exactly one solution for* $0 < \lambda < \lambda^*$.

2. Main results

2.1. Necessary and sufficient conditions for the existence of nonnegative solutions

The next three Theorems 2.1, 2.2 and 2.3 improve Theorems 1.2, 1.3 and 1.4, respectively.

Theorem 2.1. *If there exists any solutions to Problem* (1.1), (1.2) *for any* $\lambda > 0$, *then*

$$\limsup_{u\to\infty} \frac{-f(u)}{u(\ln u)^2} = \infty. \tag{2.1}$$

Theorem 2.2. *If* $f(u)$ *satisfies*

$$\liminf_{u\to\infty} \frac{-f(u)}{u(\ln u)^3} = L \quad (0 < L \le \infty), \tag{2.2}$$

then there exist solutions to Problem (1.1), (1.2) *for some* $\lambda > 0$. *Furthermore*, $G(\rho)$ *is well defined and continuous for all* $\rho \in I$.

Theorem 2.3. *If* $f(u)$ *satisfies Eq.* (2.2), *then*

$$G(\rho) \to 0^+ \quad \text{as } \rho \to \infty. \tag{2.3}$$

The next Theorems 2.4 and 2.5 improve two results of Anuradha et al. [1] slightly for more general nonlinearities $f(u)$ satisfying Eq. (2.2) instead of Eq. (1.9).

Theorem 2.4 (Cf. Anuradha et al. [1, Lemma 4.2]). *Let* $f(u)$ *satisfy Eq.* (2.2). *Assume that there exits* $s \in [0,\infty)$ *such that* $f(s)=0$. *If there exits* $\varepsilon > 0$ *such that* $(s, s+\varepsilon) \subset I$, *then* $G(\rho) \to \infty$ as $\rho \to s^+$. *Furthermore, if there exits* $\varepsilon > 0$ *such that* $(s-\varepsilon, s) \subset I$, *then* $G(\rho) \to \infty$ *as* $\rho \to s^-$.

Theorem 2.5. *Let* $f(u)$ *satisfy Eq.* (2.2) *and* $0 \in I$, *then*

$$G(0) < \infty. \tag{2.4}$$

2.2. Multiplicity results

In this section, for $G(\rho)$ in Eq. (1.7), to make it more clear for the dependence on the nonlinearity f, we write

$$G_f(\rho) := \sqrt{2} \int_\rho^\infty \frac{\mathrm{d}u}{(F(\rho) - F(u))^{1/2}} \quad \left(F(u) = \int_0^u f(t)\,\mathrm{d}t\right)$$

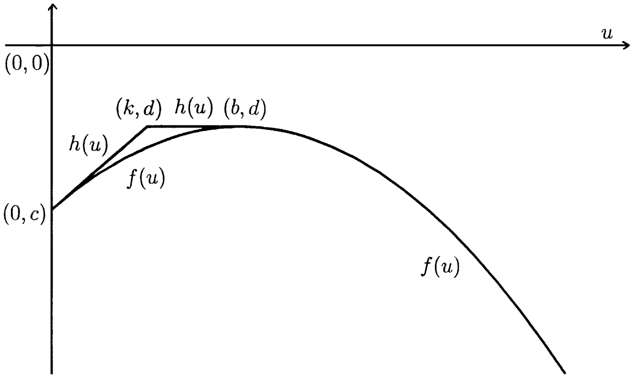

Fig. 1. Graphs of f and h.

and let

$$G_h(\rho) := \sqrt{2}\int_\rho^\infty \frac{\mathrm{d}u}{(H(\rho)-H(u))^{1/2}} \quad \left(H(u) = \int_0^u h(t)\,\mathrm{d}t\right)$$

if necessary.

Anuradha et al. [1, Theorem 3.4] implies that, if f satisfies Eq. (2.2) and is negative and nonincreasing on $(0,\infty)$, then $G_f(\rho)$ is strictly decreasing on $(0,\infty)$, and hence problem (1.1), (1.2) has at most one solution for each $\lambda > 0$. The next two theorems extend Theorem 1.5 for $f(u)$ satisfying $f(u) < 0$ on $(0,\infty)$ and is increasing on some subinterval of $(0,\infty)$. There are two cases to be considered: i.e., (i) $f(0) < 0$ and (ii) $f(0) = 0$. In Theorems 2.6 and 2.8, in addition to some conditions on f, for $f(u)$ satisfying $f(0) < 0$ and $f(0) = 0$, we show that problem (1.1), (1.2) has at least two and three nonnegative solutions for some $\lambda > 0$ respectively.

Theorem 2.6 ($f(0) < 0$, cf. Theorem 1.5(i), see Fig. 1). *In addition to Eq.* (2.2), *suppose that* $f(u)$ *satisfies* $f(u) < 0$ *on* $[0,\infty)$ *and* $F(u) = \int_0^u f(t)\,\mathrm{d}t$ *is an elementary function. Assume that there exist a function* $h(u)$ *and constants* $b > k > 0$, $c < d = f(b) < 0$ *satisfying*

$$f(u) \le h(u) := \frac{(d-c)u}{k} + c, \quad 0 \le u < k,$$

$$f(u) \le h(u) := d, \quad k \le u \le b,$$

$$f(u) \le h(u) := f(u), \quad u > b. \tag{2.5}$$

Suppose that there exist $n, m \in \aleph$ *such that*

$$T := \tilde{C}_1 - A_1 - A_2 - \tilde{B}_1 > 0, \tag{2.6}$$

where

$$\tilde{C}_1 := \frac{b}{2^n}\sum_{i=1}^{2^n}\frac{1}{(F(b)-F((i/2^n+1)b))^{1/2}}, \tag{2.7}$$

$$A_1 := \int_0^k \frac{\mathrm{d}u}{(-H(u))^{1/2}} = \sqrt{\frac{2k}{d-c}}\cos^{-1}\left(\frac{d}{c}\right), \tag{2.8}$$

$$A_2 := \int_k^b \frac{\mathrm{d}u}{(-H(u))^{1/2}} = \frac{\sqrt{2}(\sqrt{-(c+d)k}-\sqrt{-2db-ck+dk})}{d}, \tag{2.9}$$

$$\tilde{B}_1 := \frac{b}{2^m}\sum_{i=0}^{2^m-1}\frac{1}{(-H((i/2^m+1)b))^{1/2}}. \tag{2.10}$$

Then

(i) $G_f(b) > G_f(0)$ *and* $\lim_{\rho\to\infty} G_f(\rho) = 0$.

(ii) $G_f(\rho)$ *has at least one critical point, a local maximum, on* $(0,\infty)$.

(iii) *Let*

$$\hat{\lambda} = \left(\min_{\rho\in[0,b]} G_f(\rho)\right)^2 \quad \text{and} \quad \check{\lambda} = \left(\max_{\rho\in[0,\infty]} G_f(\rho)\right)^2,$$

then problem (1.1), (1.2) *has at least two nonnegative solutions for* $\hat{\lambda} < \lambda < \check{\lambda}$, *at least one nonnegative solution for* $0 < \lambda \le \hat{\lambda}$ *and* $\lambda = \check{\lambda}$, *and no nonnegative solutions for* $\lambda > \check{\lambda}$.

Note. For $\tilde{C}_1$ in Eq. (2.7),

$$\tilde{C}_1 := \frac{b}{2^n}\sum_{i=1}^{2^n}\frac{1}{(F(b)-F((i/2^n+1)b))^{1/2}}$$

is a Riemann sum for the function $(F(b)-F(u+b))^{-1/2}$ on the interval $[0,b]$ divided into 2^n even subintervals, and the sampling points are chosen to be the right end points of the subintervals. Similarly, for $\tilde{B}_1$ in Eq. (2.10),

$$\tilde{B}_1 := \frac{b}{2^m}\sum_{i=0}^{2^m-1}\frac{1}{(-H((i/2^m+1)b))^{1/2}}$$

is a Riemann sum for the function $(-H(u+b))^{-1/2}$ on the interval $[0,b]$ divided into 2^m even subintervals, and the sampling points are chosen to be the left end points of the subintervals.

The next example illustrates Theorem 2.6 for $f(u) = -(u-1)^2 + d, -0.3047 < d < 0$. This theoretical result agrees with numerical simulations obtained in Fig. 3.

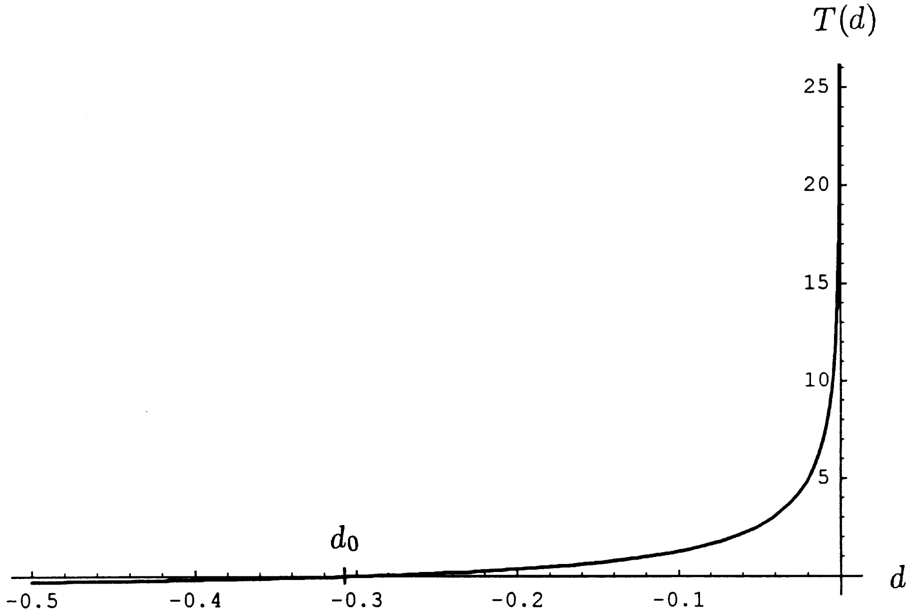

Fig. 2. $T(d)$ for $f(u)=-(u-1)^2+d,\ d<0.\ d_0\approx-0.30474$.

Example 2.7 (*See Figs.* 2 *and* 3). Let $f(u)=-(u-1)^2+d,\ d<0$. So $f(u)<0$ on $[0,\infty)$, $f(u)$ satisfies Eq. (2.2), and $F(u)=-(u-1)^3/3+\mathrm{d}u-1/3$. Choose $b=1,\ c=-1+d,\ k=1/2$. Then it is easy to see that, for $d<0$,

$$f(u)=-(u-1)^2+d\le h(u)=2u-1+d,\quad 0\le u<1/2,$$

$$f(u)=-(u-1)^2+d\le h(u)=d,\quad 1/2\le u\le 1,$$

$$f(u)=-(u-1)^2+d\le h(u)=f(u),\quad u>1.$$

So Eq. (2.5) holds. Moreover, let $n=10$ and $m=2$, then it can be computed that Eq. (2.6) holds for $-0.3047<d<0$ by using the symbolic manipulator *Mathematica*. We note that $d=d_0\approx-0.30474$ is the unique negative root of $T=T(d)$ on $(0,\infty)$. Hence, for $-0.3047<d<0$, there exist $\check{\lambda}>\hat{\lambda}>0$ such that Problem (1.1), (1.2) has at least two nonnegative solutions for $\hat{\lambda}<\lambda<\check{\lambda}$. We remark that, while for $d\le-5$, numerical simulations show uniqueness of nonnegative solution for all $\lambda>0$.

Remark 2 (See Fig. 4). *For* $f(u)=-5u^3+9u^2-3u-1+d,\ d<0$ *satisfying Eq.* (2.2), $f'(0)=-3<0$, $f(0)=-1+d<d=f(1)$ *and* $f(u)<0$ *on* $[0,\infty)$, *numerical simulations show that Problem* (1.1), (1.2) *has more than two nonnegative solutions for some* $\lambda>0$, *if* $-1<d<0$, *cf.* [1, *Fig.* 7].

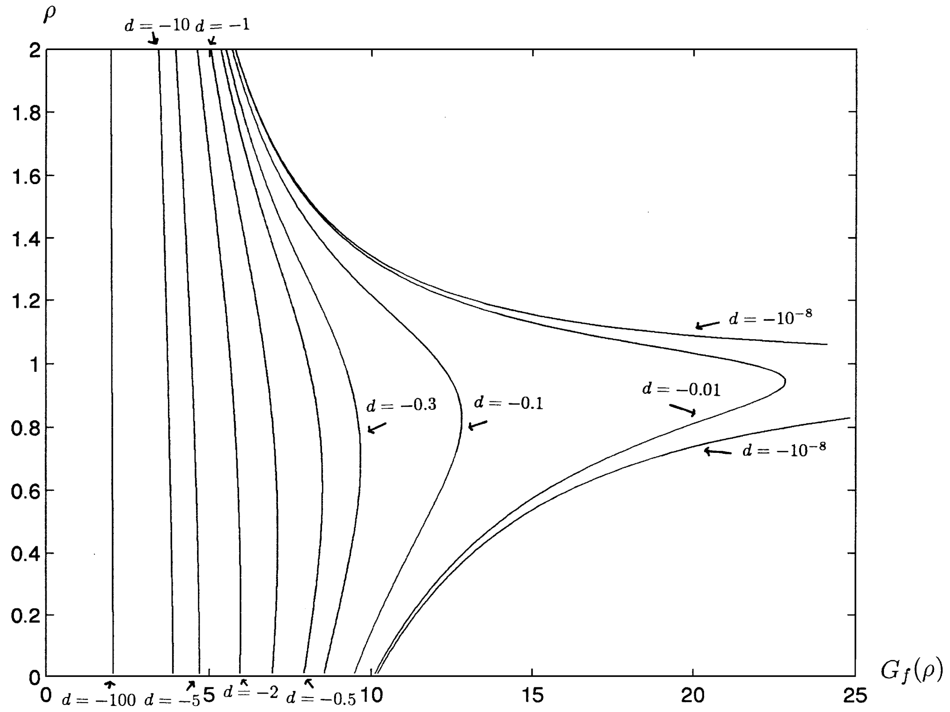

Fig. 3. Numerical simulations of $G_f(\rho)$: $f(u)=-(u-1)^2+d$, $d=-100,-10,-5,-2,-1,-0.5,-0.3,$ $-0.1,-0.01,-10^{-8}$.

Theorem 2.8 ($f(0)=0$, cf. Theorem 1.5(ii), see Fig. 5). *In addition to Eq.* (2.2), *suppose that* $f(u)$ *satisfies* $f(0)=0$ *and* $f(u)<0$ *on* $(0,\infty)$ *and* $F(u)=\int_0^u f(t)\,\mathrm{d}t$ *is an elementary function. Assume that there exist a function* $h(u)$ *and constants* $b>k>p>0,\ c=f(p)<d=f(b)<0$ *satisfying*

$$\begin{aligned}
&f(u)\le h(u):=f(u), \quad 0\le u<p,\\
&f(u)\le h(u):=\frac{(d-c)(u-p)}{k-p}+c, \quad p\le u\le k,\\
&f(u)\le h(u):=d, \quad k<u\le b,\\
&f(u)\le h(u):=f(u), \quad u>b.
\end{aligned} \tag{2.11}$$

Suppose that there exist $n,m\in\aleph$ *such that*

$$T:=\tilde{C}_1-A_1-A_2-\tilde{B}_1>0, \tag{2.12}$$

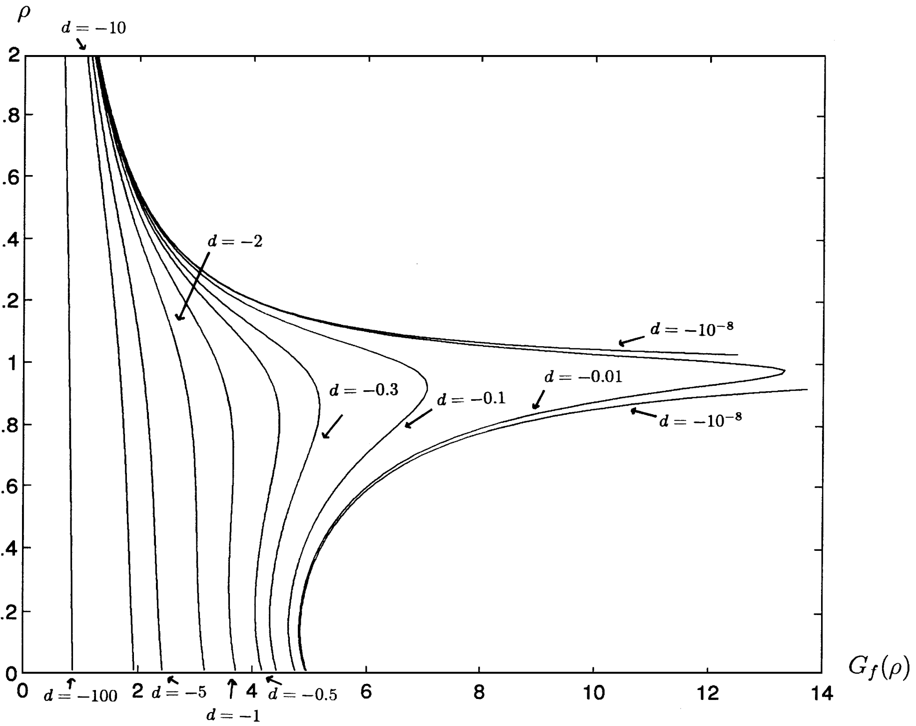

Fig. 4. Numerical simulations of $G_f(\rho)$: $f(u)=-5u^3+9u^2-3u-1+d$, $d=-100,-10,-5,-2,-1,-0.5,$ $-0.3,-0.1,-0.01,-10^{-8}$.

where

$$\tilde{C}_1 := \frac{b-p}{2^n}\sum_{i=1}^{2^n}\frac{1}{(F(b)-F((b+(i/2^n)(b-p)))^{1/2}}, \tag{2.13}$$

$$A_1 := \int_p^k \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} = \sqrt{\frac{2(k-p)}{d-c}}\cos^{-1}\left(\frac{d}{c}\right), \tag{2.14}$$

$$\begin{aligned} A_2 &:= \int_k^b \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} \\ &= \frac{\sqrt{2}(\sqrt{-(c+d)(k-p)}-\sqrt{-2d(b-p)-c(k-p)+d(k-p)})}{d}, \end{aligned} \tag{2.15}$$

$$\tilde{B}_1 := \frac{b-p}{2^m}\sum_{i=0}^{2^m-1}\frac{1}{(H(p)-H(b+(i/2^m)(b-p)))^{1/2}}. \tag{2.16}$$

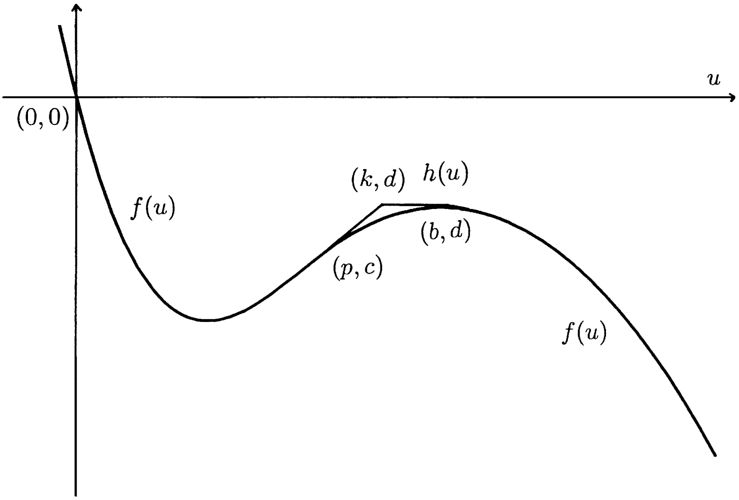

Fig. 5. Graphs of f and h.

Then

(i) $G_f(b) > G_f(p)$, $\lim_{\rho\to 0^+} G_f(\rho) = \infty$ *and* $\lim_{\rho\to\infty} G_f(\rho) = 0$.

(ii) $G_f(\rho)$ *has at least two critical points, a local minimum and a local maximum, on* $(0,\infty)$.

(iii) *Let*

$$\hat{\lambda} = \left(\min_{\rho\in[0,b]} G_f(\rho)\right)^2 \quad \textit{and} \quad \check{\lambda} = \left(\max_{\rho\in[0,\infty]} G_f(\rho)\right)^2,$$

then problem (1.1), (1.2) *has at least three nonnegative solutions for* $\hat{\lambda} < \lambda < \check{\lambda}$, *at least two nonnegative solution for* $\lambda = \hat{\lambda}$ *and* $\lambda = \check{\lambda}$, *and at least one nonnegative solution for* $0 < \lambda < \hat{\lambda}$ *and* $\lambda > \check{\lambda}$.

Note. For $\tilde{C}_1$ in Eq. (2.13),

$$\tilde{C}_1 := \frac{b-p}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b) - F((b + (i/2^n)(b-p)))^{1/2}}$$

is a Riemann sum for the function $(F(b) - F(u + b - p))^{-1/2}$ on the interval $[p, b]$ divided into 2^n even subintervals, and the sampling points are chosen to be the right end points of the subintervals. Similarly, for $\tilde{B}_1$ in Eq. (2.16),

$$\tilde{B}_1 := \frac{b-p}{2^m} \sum_{i=0}^{2^m-1} \frac{1}{(H(p) - H(b + (i/2^m)(b-p)))^{1/2}}$$

is a Riemann sum for the function $(H(p)-H(u+b-p))^{-1/2}$ on the interval $[p,b]$ divided into 2^m even subintervals, and the sampling points are chosen to be the left end points of the subintervals.

The next example illustrates Theorem 2.8 for $f(u)=-\chi(-u-r>0)(u+r)^3-(u+r-1)^2+d$, $-0.3047<d<0$, where

$$\chi(u>0):=1 \quad \text{if } u>0,$$

$$\chi(u>0):=0 \quad \text{if } u\le 0.$$

Example 2.9 (*Cf. Example* 2.7 *and see Fig.* 5)**.** Let

$$f(u)=-\chi(-u-r>0)(u+r)^3-(u+r-1)^2+d, \quad -0.3047<d<0,$$

where $r<0$ is the unique real zero of $f(u-r)=-\chi(-u>0)u^3-(u-1)^2+d$. Choose

$$\begin{aligned}&b=-r+1, \quad c=-1+d, \quad k=-r+1/2,\\ &p=-r, \quad n=10, \quad m=2.\end{aligned} \tag{2.17}$$

It can be checked that $f(u)$ satisfies all the hypotheses of Theorem 2.8 for $-0.3047<d<0$. Hence, for $-0.3047<d<0$, there exist $\check{\lambda}>\hat{\lambda}>0$ such that problem (1.1), (1.2) has at least three nonnegative solutions for $\hat{\lambda}<\lambda<\check{\lambda}$. We note that, actually, let function $g(u)$ satisfy $g(0)=0$ such that the function

$$f(u-r):=-\chi(-u>0)g(u)-(u-1)^2+d, \quad d<0$$

is continuous on $(-\infty,\infty)$ and has a unique real zero $\gamma<0$, then

$$f(u)=-\chi(-u-r>0)g(u+r)-(u+r-1)^2+d, \quad -0.3047<d<0$$

satisfies all the hypotheses of Theorem 2.8 if Eq. (2.17) is chosen.

3. Proofs of main results

The proofs of Theorems 2.1–2.5 are based upon modification of methods of Anuradha et al. [1] used to prove Theorems 1.2–1.4.

Proof of Theorem 2.1. Assume that $\limsup_{u\to\infty}(-f(u)/u(\ln u)^2)\neq\infty$. Then there exist constants $K>0$, $M_1>2$ such that

$$-f(u)\le Ku(\ln u)^2 \quad \text{for } u>M_1.$$

So

$$-f(u)\le Ku(\ln u)^2<K[u(\ln u)^2+u\ln u] \quad \text{for } u>M_1>2.$$

This implies that

$$\begin{aligned}
-F(u) &= \int_0^u -f(t)\,\mathrm{d}t,\\
&= -F(M_1) + \int_{M_1}^u -f(t)\,\mathrm{d}t\\
&< -F(M_1) + \int_{M_1}^u K[t(\ln t)^2 + t\ln t]\,\mathrm{d}t\\
&= -F(M_1) + \frac{K}{2}u^2(\ln u)^2 - \frac{K}{2}M_1^2(\ln M_1)^2 \quad \text{for } u > M_1.
\end{aligned}$$

Let $\rho \in I$, then

$$F(\rho) - F(u) < F(\rho) - F(M_1) + \frac{K}{2}u^2(\ln u)^2 - \frac{K}{2}M_1^2(\ln M_1)^2 \quad \text{for } u > M_1.$$

Let $K_1 = F(\rho) - F(M_1) - (K/2)M_1^2(\ln M_1)^2$, then

$$F(\rho) - F(u) < K_1 + \frac{K}{2}u^2(\ln u)^2 \quad \text{for } u > M_1. \tag{3.1}$$

There exists $M_2 > 2$ such that

$$\frac{K}{2}u^2(\ln u)^2 > K_1 \quad \text{for } u > M_2 > 2. \tag{3.2}$$

Let $M = \max\{M_1, M_2\}$, then Eqs. (3.1) and (3.2) imply

$$F(\rho) - F(u) < Ku^2(\ln u)^2 \quad \text{for } u > M. \tag{3.3}$$

Without loss of generality, we may assume $M > \max\{\rho, 2\}$ and obtain from Eq. (3.3) that

$$\begin{aligned}
G(\rho) &= \sqrt{2}\int_\rho^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}}\\
&\geq \sqrt{2}\int_M^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}}\\
&\geq \sqrt{2}\int_M^\infty \frac{\mathrm{d}u}{\sqrt{Ku^2(\ln u)^2}}\\
&= \frac{\sqrt{2}}{\sqrt{K}}\int_M^\infty \frac{\mathrm{d}u}{u\ln u}\\
&= \frac{\sqrt{2}}{\sqrt{K}} \ln(\ln u)\Big|_{u=M}^{u=\infty}\\
&= \infty.
\end{aligned}$$

Thus $G(\rho)$ does not exist if $\limsup_{u\to\infty}(-f(u)/u(\ln u)^2)\neq\infty$, and Theorem 2.1 follows from Lemma 1.1. □

To prove Theorems 2.2 and 2.5, we need a technical lemma which is similar to [1, Lemma 4.1].

Lemma 3.1. *Let f satisfy Eq.* (2.2) *and $\rho\in[\rho_1,\rho_2]\subset I$. Then there exists $M>e^3$ such that*

$$F(\rho)-F(u)\geq Cu^2(\ln u)^3 \quad \textit{for } u>M,\ \rho\in[\rho_1,\rho_2], \tag{3.4}$$

where

$$\begin{aligned} &C=L/12 \quad \textit{if } 0<L<\infty,\\ &C=1/6 \quad \textit{if } L=\infty. \end{aligned} \tag{3.5}$$

Proof. If f satisfies Eq. (2.2), it is easy to see that there exists a constant $M_3>e^3$ such that

$$-f(u)>6Cu(\ln u)^3>h(u):=2Cu[2(\ln u)^3+3(\ln u)^2] \quad \text{for } u>M_3>e^3, \tag{3.6}$$

where C is defined in Eq. (3.5). Then, for $u>M_3$,

$$\begin{aligned} -F(u)&=-F(M_3)+\int_{M_3}^{u}-f(t)\,\mathrm{d}t\\ &\geq -F(M_3)+\int_{M_3}^{u}h(t)\,\mathrm{d}t\\ &=-F(M_3)+2C[u^2(\ln u)^3-M_3^2(\ln M_3)^3]. \end{aligned}$$

Let $\rho\in I$. Since I is open, there exist ρ_1 and $\rho_2\in I$ and $[\rho_1,\rho_2]\subset I$, see [1, p. 623, line 10]. Let $K=-F(M_3)-2CM_3^2(\ln M_3)^3+\inf_{\rho\in[\rho_1,\rho_2]}F(\rho)$. We obtain

$$F(\rho)-F(u)\geq K+2Cu^2(\ln u)^3 \quad \text{for } u>M_3,\ \rho\in[\rho_1,\rho_2]. \tag{3.7}$$

Now there exists $M_4>2$ such that

$$Cu^2(\ln u)^3\geq -K \quad \text{for } u>M_4$$

which implies

$$K+2Cu^2(\ln u)^3\geq Cu^2(\ln u)^3 \quad \text{for } u>M_4. \tag{3.8}$$

Letting $M=\max\{M_3,M_4\}>e^3$, by Eqs. (3.7) and (3.8), we obtain

$$F(\rho)-F(u)\geq Cu^2(\ln u)^3 \quad \text{for } u>M,\ \rho\in[\rho_1,\rho_2].$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 2.2. Let $\rho \in I$. Since I is open, there exist $\rho_1, \rho_2 \in I$ and $[\rho_1, \rho_2] \subset I$. Suppose that f satisfies Eq. (2.2), by Lemma 3.1, then there exists a constant $M > \max\{e^3, \rho_2\}$ (we assume without loss of generality that $M > \max\{e^3, \rho_2\}$) such that

$$F(\rho) - F(u) \geq Cu^2(\ln u)^3 \quad \text{for } u > M,\ \rho \in [\rho_1, \rho_2], \tag{3.9}$$

where C is defined in Eq. (3.5). Note that

$$G(\rho) = \sqrt{2} \int_\rho^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} < \infty$$

if and only if there exists $\delta > 0$ such that

$$\int_\rho^{\rho+\delta} \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} < \infty, \quad 0 < \delta < \rho_2 - \rho,$$

and

$$\int_M^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} < \infty.$$

Let

$$L^* = \min_{z \in [\rho, \rho+\delta]} (-f(z)) > 0.$$

For $u \in [\rho, \rho + \delta]$, by the mean value theorem, we obtain

$$F(\rho) - F(u) = -f(z)(u - \rho) \geq \left[\min_{z \in [\rho, \rho+\delta]} (-f(z))\right](u - \rho) = L^*(u - \rho).$$

Thus

$$\int_\rho^{\rho+\delta} \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} \leq \frac{1}{\sqrt{L^*}} \int_\rho^{\rho+\delta} \frac{\mathrm{d}u}{\sqrt{u - \rho}} = \frac{2\sqrt{\delta}}{\sqrt{L^*}} < \infty. \tag{3.10}$$

Also, by Eq. (3.9), we have

$$\begin{aligned} \int_M^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} &\leq \frac{1}{\sqrt{C}} \int_M^\infty \frac{\mathrm{d}u}{u(\ln u)^{3/2}} \\ &= \frac{2}{\sqrt{C}} (\ln M)^{-1/2} \\ &< \infty. \end{aligned} \tag{3.11}$$

Eqs. (3.10) and (3.11) imply

$$G(\rho) < \infty \quad \text{for } \rho \in I_f.$$

Hence $G(\rho)$ is well defined for all $\rho \in I$, and by Lemma 1.1, there exists a solution to Problem (1.1), (1.2) for some $\lambda = [G(\rho)]^2$ given by any $\rho \in I$. By using Eq. (3.9), the same method of [1, Theorem 3.2, p. 624] can be applied to show that $G(\rho)$ is continuous for all $\rho \in I$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Suppose that f satisfies Eq. (2.2); by Theorem 2.2, $G(\rho)$ exists and is continuous on I. Moreover, in Eq. (3.6), it is known that, there exists a constant $M_3 > e^3$ such that

$$-f(u) > 6Cu(\ln u)^3 > h(u) := 2Cu[2(\ln u)^3 + 3(\ln u)^2] \quad \text{for } u > M_3, \tag{3.12}$$

where C is defined in Eq. (3.5). Thus for $u > \rho \geq M_3$,

$$F(\rho) - F(u) = \int_\rho^u -f(t)\,\mathrm{d}t \geq \int_\rho^u h(t)\,\mathrm{d}t = 2C[u^2(\ln u)^3 - \rho^2(\ln \rho)^3].$$

If we let $\rho^* = 2\rho$, then for $u \geq \rho^* = 2\rho$, we have

$$F(\rho) - F(u) \geq 2C[u^2(\ln u)^3 - \rho^2(\ln \rho)^3] \geq (3C/2)u^2(\ln u)^3 \tag{3.13}$$

since

$$u^2(\ln u)^3 > 4\rho^2(\ln \rho)^3.$$

Then, by Eq. (3.13), for $\rho \geq M_3 > e^3$,

$$\begin{aligned} G(\rho) &= \sqrt{2}\int_\rho^\infty \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} \\ &\leq \sqrt{2}\left\{\int_\rho^{\rho^*} \frac{\mathrm{d}u}{\sqrt{F(\rho) - F(u)}} + \int_{\rho^*}^\infty \frac{\sqrt{2}}{\sqrt{3C}}\frac{\mathrm{d}u}{u(\ln u)^{3/2}}\right\} \\ &= \sqrt{2}\left\{\int_\rho^{\rho^*} \frac{\mathrm{d}u}{\sqrt{-f(z)}\sqrt{u-\rho}} + \frac{2\sqrt{2}}{\sqrt{3C}(\ln \rho^*)^{1/2}}\right\}, \end{aligned}$$

where $z = z(u) \in (\rho, u)$ for each $u \in (\rho, \rho^*)$ exists by the mean value theorem. However, Eq. (3.12) implies

$$-f(z) \geq 4Cz(\ln z)^3 > 4C\rho(\ln \rho)^3 \quad \text{for } z > \rho \geq M_3.$$

So we have

$$\begin{aligned} G(\rho) &\leq \sqrt{2}\left\{\int_\rho^{\rho^*} \frac{\mathrm{d}u}{\sqrt{-f(z)}\sqrt{u-\rho}} + \frac{2\sqrt{2}}{\sqrt{3C}(\ln \rho^*)^{1/2}}\right\} \\ &< \sqrt{2}\left\{\frac{1}{2\sqrt{C}\rho^{1/2}(\ln \rho)^{3/2}}\int_\rho^{\rho^*} \frac{\mathrm{d}u}{\sqrt{u-\rho}} + \frac{2\sqrt{2}}{\sqrt{3C}(\ln \rho^*)^{1/2}}\right\} \end{aligned}$$

$$= \sqrt{2}\left\{\frac{1}{\sqrt{C}(\ln\rho)^{3/2}} + \frac{2\sqrt{2}}{\sqrt{3C}(\ln 2\rho)^{1/2}}\right\} \quad (\text{since } \rho^* = 2\rho)$$

$$\to 0 \quad \text{as } \rho \to \infty.$$

This completes the proof of Theorem 2.3. □

The proof of Theorem 2.4 is exactly the same as that of [1, Lemma 4.2], and hence is omitted.

The proof of Theorem 2.5 is quite similar to that of Theorem 2.2.

Proof of Theorem 2.5. Let $0 \in I$, then there exists $\rho_2 \in I$ such that $[0, \rho_2] \in I$. It is well known that

$$G(0) = \sqrt{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{-F(u)}} < \infty$$

if and only if there exist numbers $0 < \delta < M$ such that

$$\int_0^\delta \frac{\mathrm{d}u}{\sqrt{-F(u)}} < \infty, \quad 0 < \delta < \rho_2,$$

and

$$\int_M^\infty \frac{\mathrm{d}u}{\sqrt{-F(u)}} < \infty, \quad M > 2.$$

Now $[0, \rho_2] \in I$ implies $L^* := \min_{z\in[0,\delta]}(-f(z)) > 0$. For $u \in [0, \delta]$, by the mean value theorem, we have

$$\begin{aligned} -F(u) &= F(0) - F(u) \\ &\geq \left(\min_{z\in[0,\delta]}(-f(z))\right)(u-0) \\ &= L^* u. \end{aligned}$$

Thus

$$\int_0^\delta \frac{\mathrm{d}u}{\sqrt{-F(u)}} \leq \frac{1}{\sqrt{L^*}}\int_0^\delta \frac{\mathrm{d}u}{\sqrt{u}} = \frac{2\sqrt{\delta}}{\sqrt{L^*}} < \infty. \tag{3.14}$$

Also, if f satisfies Eq. (2.2), by Lemma 3.1 with $\rho = 0$, it is known that there exists $M > 2$ such that

$$-F(u) \geq Cu^2(\ln u)^3 \quad \text{for } u > M > 2.$$

Thus

$$\int_M^\infty \frac{\mathrm{d}u}{\sqrt{-F(u)}} \le \frac{1}{\sqrt{C}} \int_M^\infty \frac{\mathrm{d}u}{u(\ln u)^{3/2}} \\ = \frac{2}{\sqrt{C}}(\ln M)^{-1/2} \\ < \infty. \tag{3.15}$$

Eqs. (3.14) and (3.15) imply $G(0) < \infty$. □

To prove Theorem 2.6, by Theorems 2.2, 2.3 and 2.5, it suffices to show

$$G_f(b) > G_f(0).$$

Proof of Theorem 2.6. Suppose that f satisfies Eq. (2.2); then h defined in Eq. (2.5) also satisfies Eq. (2.2). By Theorem 2.2, both functions $G_f(\rho)$ and $G_h(\rho)$ exist and are continuous on $[0,\infty)$. Recall $A_1, A_2, \tilde{B}_1, \tilde{C}_1$ in Eqs. (2.7)–(2.10) and define B_1, B_2, C_1, C_2 as follows:

$$A_1 := \int_0^k \frac{\mathrm{d}u}{(-H(u))^{1/2}} = \sqrt{\frac{2k}{d-c}} \cos^{-1}\left(\frac{d}{c}\right) \quad \text{(after simple calculation)}, \tag{3.16}$$

$$A_2 := \int_k^b \frac{\mathrm{d}u}{(-H(u))^{1/2}} \\ = \frac{\sqrt{2}(\sqrt{-(c+d)k} - \sqrt{-2db - ck + dk})}{d} \\ \text{(after simple calculation)}, \tag{3.17}$$

$$B_1 := \int_0^b \frac{\mathrm{d}u}{(-H(u+b))^{1/2}}, \tag{3.18}$$

$$B_2 := \int_b^\infty \frac{\mathrm{d}u}{(-H(u+b))^{1/2}}, \tag{3.19}$$

$$C_1 := \int_0^b \frac{\mathrm{d}u}{(F(b) - F(u+b))^{1/2}}, \tag{3.20}$$

$$C_2 := \int_b^\infty \frac{\mathrm{d}u}{(F(b) - F(u+b))^{1/2}}, \tag{3.21}$$

$$\tilde{B}_1 := \frac{b}{2^m} \sum_{i=0}^{2^m-1} \frac{1}{(-H((i/2^m+1)b))^{1/2}} \quad (m \geq 1), \tag{3.22}$$

$$\tilde{C}_1 := \frac{b}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b)-F((i/2^n+1)b))^{1/2}} \quad (n \geq 1). \tag{3.23}$$

(i) For C_1 in Eq. (3.20), the integrand of the integral

$$Int1(u) := \frac{1}{(F(b)-F(u+b))^{1/2}} > 0 \quad \text{for } 0 < u \leq b$$

and satisfies

$$\lim_{u \to 0^+} Int1(u) = \infty$$

and

$$Int1(u) \text{ is a strictly decreasing function on } (0,b),$$

since $f(u) < 0$ on $[0,\infty)$. Hence

$$\begin{aligned} C_1 &= \int_0^b \frac{\mathrm{d}u}{(F(b)-F(u+b))^{1/2}} \\ &> \frac{b}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b)-F(((i/2^n)+1)b))^{1/2}} = \tilde{C}_1. \end{aligned} \tag{3.24}$$

(ii) For B_1 in Eq. (3.18), similarly, the integrand of the integral

$$Int2(u) := \frac{1}{(-H(u+b))^{1/2}} > 0 \quad \text{for } 0 \leq u \leq b$$

and satisfies

$$Int2(0) = \frac{1}{(-H(b))^{1/2}} > 0$$

and

$$Int2(u) \text{ is a strictly decreasing function on } (0,b).$$

Hence

$$B_1 = \int_0^b \frac{\mathrm{d}u}{(-H(u+b))^{1/2}} < \frac{b}{2^m} \sum_{i=0}^{2^m-1} \frac{1}{(-H((i/2^m+1)b))^{1/2}} = \tilde{B}_1. \tag{3.25}$$

(iii) For B_2 in Eq. (3.19), it is easy to see that

$$\begin{aligned}
B_2 &= \int_b^\infty \frac{\mathrm{d}u}{\{-H(u+b)\}^{1/2}} \\
&= \int_b^\infty \frac{\mathrm{d}u}{\{[H(b)-H(u+b)]-H(b)\}^{1/2}} \\
&< \int_b^\infty \frac{\mathrm{d}u}{\{H(b)-H(u+b)\}^{1/2}} \quad (\text{since } -H(b)>0) \\
&= \int_b^\infty \frac{\mathrm{d}u}{\{\int_b^{u+b} -h(t)\,\mathrm{d}t)\}^{1/2}} \\
&= \int_b^\infty \frac{\mathrm{d}u}{\{\int_b^{u+b} -f(t)\,\mathrm{d}t)\}^{1/2}} \quad (\text{since } h(t)=f(t) \text{ on } (b,\infty)) \\
&= \int_b^\infty \frac{\mathrm{d}u}{\{F(b)-F(u+b)\}^{1/2}} \\
&= C_2.
\end{aligned} \tag{3.26}$$

(iv) By Eqs. (3.20), (3.21) and (3.24), we obtain

$$\begin{aligned}
2^{-1/2}G_f(b) &= \int_b^\infty \frac{\mathrm{d}u}{(F(b)-F(u))^{1/2}} \\
&= \int_0^\infty \frac{\mathrm{d}y}{(F(b)-F(y+b))^{1/2}} \quad (\text{by setting } u=y+b) \\
&= \int_0^b \frac{\mathrm{d}y}{(F(b)-F(y+b))^{1/2}} + \int_b^\infty \frac{\mathrm{d}y}{(F(b)-F(y+b))^{1/2}} \\
&= C_1 + C_2 \\
&> \tilde{C}_1 + C_2.
\end{aligned} \tag{3.27}$$

(v) By Eqs. (2.5) and (3.16)–(3.19), we obtain

$$\begin{aligned}
2^{-1/2}G_f(0) &= \int_0^\infty \frac{\mathrm{d}u}{(-F(u))^{1/2}} \\
&\le \int_0^\infty \frac{\mathrm{d}u}{(-H(u))^{1/2}} \\
&= \int_0^b \frac{\mathrm{d}u}{(-H(u))^{1/2}} + \int_b^\infty \frac{\mathrm{d}u}{(-H(u))^{1/2}}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^b \frac{\mathrm{d}u}{(-H(u))^{1/2}} + \int_0^\infty \frac{\mathrm{d}y}{(-H(y+b))^{1/2}} \quad \text{(by setting } u=y+b\text{)}\\
&= \int_0^k \frac{\mathrm{d}u}{(-H(u))^{1/2}} + \int_k^b \frac{\mathrm{d}u}{(-H(u))^{1/2}} + \int_0^b \frac{\mathrm{d}y}{(-H(y+b))^{1/2}}\\
&\quad + \int_b^\infty \frac{\mathrm{d}y}{(-H(y+b))^{1/2}}\\
&= A_1 + A_2 + B_1 + B_2. \qquad (3.28)
\end{aligned}
$$

Thus, by Eqs. (3.25)–(3.28) and (2.6), it follows that

$$
\begin{aligned}
2^{-1/2}[G_f(b) - G_f(0)] &> \tilde{C}_1 + C_2 - (A_1 + A_2 + B_1 + B_2)\\
&> \tilde{C}_1 - A_1 - A_2 - \tilde{B}_1\\
&> 0.
\end{aligned}
$$

So $G_f(b) > G_f(0)$. By Theorem 2.3, $\lim_{\rho\to\infty} G_f(\rho) = 0$. Hence $G_f(\rho)$ has at least one critical point, a local maximum, on $(0,\infty)$. Let $\hat{\lambda} = (\min_{\rho\in[0,b]} G_f(\rho))^2$ and $\check{\lambda} = (\max_{\rho\in[0,\infty]} G_f(\rho))^2$, then it follows by Lemma 1.1 that problem (1.1), (1.2) has at least two nonnegative solutions for $\hat{\lambda} < \lambda < \check{\lambda}$, at least one nonnegative solution for $0 < \lambda \le \hat{\lambda}$ and $\lambda = \check{\lambda}$, and no nonnegative solutions for $\lambda > \check{\lambda}$. □

The proof of Theorem 2.8 is slight modification of that of Theorem 2.6. By Theorems 2.2–2.4, it suffices to show

$$
G_f(b) > G_f(p). \qquad (3.29)
$$

Proof of Theorem 2.8. Suppose that f satisfies Eq. (2.2), then h defined in Eq. (2.11) also satisfies Eq. (2.2). By Theorem 2.2, both functions $G_f(\rho)$ and $G_h(\rho)$ exist and are continuous on $(0,\infty)$. Recall A_1, A_2, $\tilde{B}_1$, $\tilde{C}_1$ in Eqs. (2.13)–(2.16) and define B_1, B_2, C_1, C_2 as follows:

$$
A_1 := \int_p^k \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} = \sqrt{\frac{2(k-p)}{d-c}}\cos^{-1}\left(\frac{d}{c}\right)
$$

(after simple calculation), (3.30)

$$
\begin{aligned}
A_2 &:= \int_k^b \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}},\\
&= \frac{\sqrt{2}\left(\sqrt{-(c+d)(k-p)} - \sqrt{-2d(b-p) - c(k-p) + d(k-p)}\right)}{d}
\end{aligned}
$$

(after simple calculation) (3.31)

$$B_1 := \int_p^b \frac{\mathrm{d}u}{(H(p) - H(u+b-p))^{1/2}}, \tag{3.32}$$

$$B_2 := \int_b^\infty \frac{\mathrm{d}u}{(H(p) - H(u+b-p))^{1/2}}, \tag{3.33}$$

$$C_1 := \int_p^b \frac{\mathrm{d}u}{(F(b) - F(u+b-p))^{1/2}}, \tag{3.34}$$

$$C_2 := \int_b^\infty \frac{\mathrm{d}u}{(F(b) - F(u+b-p))^{1/2}}, \tag{3.35}$$

$$\tilde{B}_1 := \frac{b-p}{2^m} \sum_{i=0}^{2^m-1} \frac{1}{(H(p) - H(b+(i/2^m)(b-p)))^{1/2}} \quad (m \geq 1), \tag{3.36}$$

$$\tilde{C}_1 := \frac{b-p}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b) - F((b+(i/2^n)(b-p)))^{1/2}} \quad (n \geq 1). \tag{3.37}$$

(i) For C_1 in Eq. (3.34), the integrand of the integral

$$Int3(u) := \frac{1}{(F(b) - F(u+b-p))^{1/2}} > 0 \quad \text{for } p < u \leq b$$

and satisfies

$$\lim_{u \to p^+} Int3(u) = \infty$$

and

$$Int3(u) \text{ is a strictly decreasing function on } (p, b),$$

since $f(u) < 0$ on $(0, \infty)$. Hence

$$\begin{aligned} C_1 &= \int_p^b \frac{\mathrm{d}u}{(F(b) - F(u+b-p))^{1/2}} \\ &> \frac{b-p}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b) - F((b+(i/2^n)(b-p)))^{1/2}} \\ &= \tilde{C}_1. \end{aligned} \tag{3.38}$$

(ii) For B_1 in Eq. (3.32), similarly, the integrand of the integral

$$Int4(u) := \frac{1}{(H(p) - H(u+b-p))^{1/2}} > 0 \quad \text{for } p \leq u \leq b$$

and satisfies

$$Int4(p) = \frac{1}{(H(p) - H(b))^{1/2}} > 0$$

and

$Int4(u)$ is a strictly decreasing function on (p, b).

Hence

$$\begin{aligned} B_1 &= \int_p^b \frac{\mathrm{d}u}{(H(p) - H(u+b-p))^{1/2}} \\ &< \frac{b-p}{2^m} \sum_{i=0}^{2^m-1} \frac{1}{(H(p) - H(b + (i/2^m)(b-p)))^{1/2}} \\ &= \tilde{B}_1. \end{aligned} \tag{3.39}$$

(iii) For B_2 in Eq. (3.33), it is easy to see that

$$\begin{aligned} B_2 &= \int_b^\infty \frac{\mathrm{d}u}{\{H(p) - H(u+b-p)\}^{1/2}} \\ &= \int_b^\infty \frac{\mathrm{d}u}{\{[H(b) - H(u+b-p)] + (H(p) - H(b))\}^{1/2}} \\ &< \int_b^\infty \frac{\mathrm{d}u}{\{H(b) - H(u+b-p)\}^{1/2}} \quad (\text{since } H(p) - H(b) > 0) \\ &= \int_b^\infty \frac{\mathrm{d}u}{\{\int_b^{u+b-p} -h(t)\mathrm{d}t)\}^{1/2}} \\ &= \int_b^\infty \frac{\mathrm{d}u}{\{\int_b^{u+b-p} -f(t)\mathrm{d}t)\}^{1/2}} \quad (\text{since } h(t) = f(t) \text{ on } (b, \infty)) \\ &= \int_b^\infty \frac{\mathrm{d}u}{\{F(b) - F(u+b-p)\}^{1/2}} \\ &= C_2. \end{aligned} \tag{3.40}$$

(iv) By (3.34), (3.35) and (3.38), we obtain

$$\begin{aligned} 2^{-1/2} G_f(b) &= \int_b^\infty \frac{\mathrm{d}u}{(F(b) - F(u))^{1/2}} \\ &= \int_p^\infty \frac{\mathrm{d}y}{(F(b) - F(y+b-p))^{1/2}} \quad (\text{by setting } u = y + b - p) \\ &= \int_p^b \frac{\mathrm{d}y}{(F(b) - F(y+b-p))^{1/2}} + \int_b^\infty \frac{\mathrm{d}y}{(F(b) - F(y+b-p))^{1/2}} \end{aligned}$$

$$\begin{aligned}&= C_1 + C_2\\ &> \tilde{C}_1 + C_2. \qquad (3.41)\end{aligned}$$

(v) By Eqs. (2.11) and (3.30)–(3.33), we obtain

$$\begin{aligned}
2^{-1/2}G_f(p) &= \int_p^\infty \frac{\mathrm{d}u}{(F(p)-F(u))^{1/2}}\\
&\le \int_p^\infty \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}}\\
&= \int_p^b \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} + \int_b^\infty \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}}\\
&= \int_p^b \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} + \int_p^\infty \frac{\mathrm{d}y}{(H(p)-H(y+b-p))^{1/2}}\\
&\qquad \text{(by setting } u=y+b-p\text{)}\\
&= \int_p^k \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}} + \int_k^b \frac{\mathrm{d}u}{(H(p)-H(u))^{1/2}}\\
&\quad + \int_p^b \frac{\mathrm{d}y}{(H(p)-H(y+b-p))^{1/2}}\\
&\quad + \int_b^\infty \frac{\mathrm{d}y}{(H(p)-H(y+b-p))^{1/2}}\\
&= A_1 + A_2 + B_1 + B_2. \qquad (3.42)
\end{aligned}$$

Thus, by Eqs. (3.39)–(3.42) and (2.12), it follows that

$$\begin{aligned}
2^{-1/2}[G_f(b) - G_f(p)] &> \tilde{C}_1 + C_2 - (A_1 + A_2 + B_1 + B_2)\\
&> \tilde{C}_1 - A_1 - A_2 - \tilde{B}_1\\
&> 0.
\end{aligned}$$

So $G_f(b) > G_f(p)$. By Theorems 2.3 and 2.4, $\lim_{\rho\to 0^+} G_f(\rho) = \infty$ and $\lim_{\rho\to\infty} G_f(\rho) = 0$. Hence $G_f(\rho)$ has at least two critical points, a local minimum and a local maximum, on $(0,\infty)$. Let $\hat{\lambda} = (\min_{\rho\in[0,b]} G_f(\rho))^2$ and $\check{\lambda} = (\max_{\rho\in[0,\infty]} G_f(\rho))^2$, then it follows by Lemma 1.1 that problem (1.1), (1.2) has at least three nonnegative solutions for $\hat{\lambda} < \lambda < \check{\lambda}$, at least two nonnegative solution for $\lambda = \hat{\lambda}$ and $\lambda = \check{\lambda}$, and at least one nonnegative solution for $0 < \lambda < \hat{\lambda}$ and $\lambda > \check{\lambda}$. □

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