CONCENTRATION PHENOMENA IN A NONLOCAL QUASILINEAR PROBLEM MODELLING PHYTOPLANKTON II: LIMITING PROFILE

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Abstract. We study the positive steady-state of a quasilinear reaction-diffusion system in one space dimension introduced by Klausmeier and Litchman for the modelling of the distributions of phytoplankton biomass and its nutrient. The system has nonlocal dependence on the biomass function, and has a biomass dependent drifting term describing the active movement of the biomass towards location of better growth condition. In the separate part I of this research, we have obtained existence and nonexistence results. In part II here, we obtain complete descriptions of the profile of the solutions when the coefficient of the drifting term is large, rigorously proving the numerically observed phenomenon of concentration of biomass for this model. Our results reveal four critical numbers for the model not observed before, and offer several further insights to the problem being modelled.

1. Introduction

We continue our investigation in [DH] on the problem

$$
(1.1) \qquad \begin{cases} -[d_1 u_x + \sigma c(x) u]_x = [g(x) - m]u, & 0 < x < 1, \\ -d_2 v_{xx} = -g(x) u, & 0 < x < 1, \\ d_1 u_x + \sigma c(x) u = 0, & x = 0,\ 1, \\ v_x(0) = 0,\ v_x(1) = \beta[v_0 - v(1)], & \end{cases}
$$

where d_1, d_2, σ, m, v_0 and β are positive constants,

$$
g(x) = f(\min\{\alpha v(x), w(x)\}),\ f(s) = \frac{rs}{K_I + s}
$$

and

$$
w(x) = w_0 \exp[-A_0 x - A \int_0^x u(s) ds],
$$

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with α, r, K_I, w_0, A and A_0 positive constants. We are interested in positive solutions of (1.1), namely $u > 0$ and $v > 0$ in $[0,1]$. From (1.1) it is easy to see that for any such solution, v is an increasing function. Clearly w is a decreasing function. The function $c(x)$ is defined by

$$c(x) = \frac{x - x_0}{\delta + |x - x_0|},$$

where $\delta > 0$ is a small constant and $x_0 \in [0,1]$ is determined by the following description:

$$\min\{\alpha v(x), w(x)\} = \alpha v(x) \ \forall x \in [0, x_0); \ \min\{\alpha v(x), w(x)\} = w(x) \ \forall x \in (x_0, 1].$$

Such a system arises in the mathematical modelling of phytoplankton in a one dimensional water column, and the term $\sigma c(x)$ is used to describe the active movement of the biomass towards spatial location with better growth condition. Klausmeier and Litchman [KL] propose to use this model to study the concentration phenomenon widely observed for phytoplankton in lakes and oceans. The numerical analysis in [KL] demonstrates that for large σ, the biomass function $u(x)$ concentrates at a certain level $x = x_*$ while the nutrient function $v(x)$ is close to a piecewise linear function. They then treat u as a constant multiple of the δ-function concentrating at x_* and propose a game theoretical model to determine the location of x_*. We refer to part I [DH] for further details regarding the background of (1.1).

Here, we rigorously prove the existence of such a concentration phenomenon, and obtain exact formulas for the determination of x_* and the total biomass. In doing so, we reveal the existence of four critical values $v_{**} < v_* < v^* < v^{**}$ for v_0 (the nutrient level at the sediment), such that

(i) $x_* = 0$ when $v_0 \geq v^*$, $x_* \in (0,1)$ when $v_0 \in (v_*, v^*)$, and $x_* = 1$ when $v_0 \leq v_*$;

(ii) the total biomass increases with v_0 in the range $v_{**} < v_0 < v^{**}$, but it stays constant for $v_0 \geq v^{**}$ or $v_0 \leq v_{**}$ (and with v_0 above a certain level so that the biomass can survive).

It turns out that the game theoretical model of [KL] is a simplified version of our limiting equations for the case $v_* \leq v_0 \leq v^*$.

In part I ([DH]), we have proved the following two theorems:

Theorem 1.1. *There exist $0 < m_* \leq m^* < \infty$ such that (1.1) has a positive solution for $m \in (0, m_*)$, and it has no positive solution for $m > m^*$.*

The values of m_* and m^* depend on the parameters in (1.1). To stress their dependence on σ, we write $m_* = m_*(\sigma)$, $m^* = m^*(\sigma)$.

Theorem 1.2.

$$\lim_{\sigma\to\infty} m_*(\sigma) = \lim_{\sigma\to\infty} m^*(\sigma) = f(\min\{\alpha v_0, w_0\}).$$

To investigate the limiting profile of the positive solutions of (1.1) as $\sigma \to \infty$, we will fix m such that $0 < m < f(\min\{\alpha v_0, w_0\})$ and let σ_n be an increasing sequence of positive numbers converging to ∞. By Theorems 1.1 and 1.2, for all large n, (1.1) with $\sigma = \sigma_n$ has at least one positive solution. Suppose that (u_n, v_n) is such a solution. We will analyze the behavior of (u_n, v_n) as $n \to \infty$. This will be done in the following two sections.

In section 2, we will find all the possible limiting profiles a subsequence of $\{(u_n, v_n)\}$ can have, in particular, we will find the limiting equations governing these possible limiting profiles. In section 3, we will show that the limiting equations obtained in section 2 have a unique solution and hence the original entire sequence $\{(u_n, v_n)\}$ has a unique limiting profile as $n \to \infty$.

Our main results here are Theorems 3.1, 3.2 and 3.3. The biological predictions of our results, and comparison of our rigorous limiting equations with the game theoretical model of [KL] are given in Remark 3.4 at the end of the paper. Though the proofs here are rather involved, they consist of mainly standard elliptic regularity arguments and elementary mathematical analysis.

2. The limiting equations

We will keep using the notations of part I [DH]. It turns out that the techniques used in the proof of Theorem 3.1 in part I are not good enough for our purpose here. We will introduce some different ones.

Suppose that $0 < m < f(\min\{\alpha v_0, w_0\})$, and σ_n, (u_n, v_n) are as given in the introduction above. Suppose $c_{v_n,w_n}(x) = C_{x_n}(x)$, $x_n \in [0,1]$. By passing to a subsequence we may assume that $x_n \to x_* \in [0,1]$. Then

$$C_{x_n} = \frac{x - x_n}{\delta + |x - x_n|} \to C_{x_*}$$

in $C^1([0,1])$. We now define

$$\Phi_n(x) = \exp\big[-\frac{\sigma_n}{2d_1}\int_{x_n}^{x} C_{x_n}(s)ds\big],$$

and

$$\Psi_n(x) = u_n(x)/\Phi_n(x).$$

By a direct computation we obtain

$$\begin{cases} -d_1\Psi_n'' + \sigma_n\Gamma_n(x)\Psi_n = \big[f(\min\{\alpha v_n, w_n\}) - m\big]\Psi_n, & x \in (0,1), \\ d_1\Psi_n' + (\sigma_n/2)C_{x_n}\Psi_n = 0, & x = 0, 1, \end{cases}$$

where

$$\Gamma_n(x) := \frac{\sigma_n(x - x_n)^2 - 2d_1\delta}{4d_1(\delta + |x - x_n|)^2}.$$

Let

$$V_n(y) := \Psi_n\big(\sigma_n^{-1/2}y + x_n\big),\ C_n(y) := \sigma_n^{1/2}C_{x_n}\big(\sigma_n^{-1/2}y + x_n\big) = \frac{y}{\delta + \sigma_n^{-1/2}|y|},$$

$$a_n := -\sigma_n^{1/2}x_n,\ b_n := \sigma_n^{1/2}(1 - x_n),$$

and

$$F_n(y) := f\big(\min\{\alpha v_n(\sigma_n^{-1/2}y + x_n), w_n(\sigma_n^{-1/2}y + x_n)\}\big).$$

Then

$$\begin{cases} -d_1V_n'' + \dfrac{y^2 - 2d_1\delta}{4d_1(\delta + \sigma_n^{-1/2}|y|)^2}V_n = \sigma_n^{-1}\big[F_n(y) - m\big]V_n, & y \in (a_n, b_n), \\ d_1V_n' + (1/2)C_nV_n = 0, & y = a_n,\ b_n. \end{cases} \tag{2.1}$$

In the discussions below, we will consider the cases $x_* \in (0,1)$, $x_* = 0$ and $x_* = 1$ separately.

Lemma 2.1. *Suppose $x_n \to x_* \in (0,1)$ and set $\tilde V_n(y) = V_n(y)/\|V_n\|_{L^\infty([a_n,b_n])}$. Then*

$$\tilde V_n \to V_0 \text{ in } C^1(J) \text{ for any finite interval } J \subset (-\infty,\infty),$$

where $V_0(y) = \exp[-\frac{y^2}{4d_1\delta}]$ is the unique solution of

$$-d_1V'' = \frac{2d_1\delta - y^2}{4d_1\delta^2}V,\ 0 < V \le 1,\ V(0) = 1,\ V'(0) = 0.$$

Proof. Since $x_* \in (0,1)$, we have $a_n \to -\infty$ and $b_n \to \infty$ as $n \to \infty$. Let us note that, for $y \in [a_n, -(2d_1\delta)^{1/2} - \epsilon]$ and all large n, the first equation in (2.1) implies that $V_n''(y) > 0$. Since $d_1V_n'(a_n) = -(1/2)C_n(a_n)V_n(a_n) \ge 0$, we deduce that $V_n'(y) > 0$ in $(a_n, -(2d_1\delta)^{1/2} - \epsilon]$ for all large n. Hence V_n is increasing in this range. Similarly, we can see that $V_n(y)$ is decreasing in the range $y \in [(2d_1\delta)^{1/2} + \epsilon, b_n]$ for all large n. Therefore $\max V_n = V_n(y_n)$ for some $y_n \in [-2d_1\delta)^{1/2} - \epsilon, (2d_1\delta)^{1/2} + \epsilon]$, and $\tilde V_n(y) = V_n(y)/V_n(y_n)$. We may assume that $y_n \to y^*$ as $n \to \infty$. We now define

$$\tilde F_n(y) := \frac{2d_1\delta - y^2}{4d_1(\delta + \sigma_n^{-1/2}|y|)^2} + \sigma_n^{-1}\big[F_n(y) - m\big].$$

Then $\tilde V_n(y_n) = 1$ and

$$\begin{cases} -d_1\tilde V_n'' = \tilde F_n\tilde V_n,\ 0 < \tilde V_n \le 1, & y \in (a_n, b_n), \\ d_1\tilde V_n' + (1/2)C_n\tilde V_n = 0, & y = a_n,\ b_n. \end{cases} \tag{2.2}$$

Since $\{\tilde F_n\}$ is uniformly bounded over any bounded interval and $0 \le \tilde V_n \le 1$, we may apply the interior L^p theory (see [GT]) to (2.2), use the Sobolev imbedding theorem and a standard diagonal argument to conclude that, by passing to a subsequence, $\tilde{V}_n \to \tilde{V}$ in $C^1(J)$ for any bounded interval J, and $\tilde{V}$ satisfies

$$-d_1\tilde{V}'' = \frac{2d_1\delta - y^2}{4d_1\delta^2}\tilde{V},\ 0 < \tilde{V} \le 1 \text{ in } (-\infty,\infty),\ \tilde{V}(y^*) = 1,\ \tilde{V}'(y^*) = 0. \tag{2.3}$$

By the monotonicity property of $V_n(y)$ observed earlier, we know that $\tilde{V}(y)$ is non-decreasing in $(-\infty, -(2d_1\delta)^{1/2})$, and is non-increasing in $((2d_1\delta)^{1/2}, \infty)$. We can now use (2.3) to conclude that $\tilde{V}'(y)$ is positive and increasing in $(-\infty, -(2d_1\delta)^{1/2})$, reaching a positive maximum at $y = -(2d_1\delta)^{1/2}$, and then is decreasing in $(-(2d_1\delta)^{1/2}, (2d_1\delta)^{1/2})$, reaching a negative minimum at $y = (2d_1\delta)^{1/2}$, and for $y > (2d_1\delta)^{1/2}$, it is increasing and stays negative. Therefore $V'(y)$ has a unique zero at some $y_0 \in \big(-(2d_1\delta)^{1/2}, (2d_1\delta)^{1/2}\big)$, which is the unique maximum point of $\tilde{V}$. Thus $y_0 = y^*$. In other words, $\tilde{V}(y)$ is increasing in $(-\infty, y^*)$ and is decreasing in $(y^*, 0)$. It then follows from an elementary analysis that $\tilde{V}$ decays to 0 as $|y| \to \infty$, and there exists $C_1, C_2 > 0$ such that

$$\tilde{V}(y),\ |\tilde{V}'(y)| \le C_1 e^{-C_2|y|}\ \ \forall y \in (-\infty,\infty).$$

We now multiply $\tilde{V}(-y)$ to (2.3), integrate over $[y^*, \infty)$ and then apply integration by parts. Since $\tilde{V}(-y)$ satisfies the differential equation in (2.3), we deduce

$$\tilde{V}'(-y^*)\tilde{V}(y^*) + \tilde{V}'(y^*)\tilde{V}(-y^*) = 0.$$

It follows that $\tilde{V}'(-y^*) = 0$. Since y^* is the only zero of $\tilde{V}'$, we must have $y^* = -y^*$, that is, $y^* = 0$. By the uniqueness theorem of initial value problems of ordinary differential equations, we must have $\tilde{V} = V_0$, the unique solution of (2.3) with $y^* = 0$. A simple calculation confirms that the function $\exp[-\frac{y^2}{4d_1\delta}]$ solves the equation for V_0. Hence, by uniqueness,

$$V_0(y) = \exp\big[-\frac{y^2}{4d_1\delta}\big].$$

Since V_0 is uniquely determined, the entire original sequence $\{\tilde{V}_n\}$ converges to V_0. □

Using the monotonicity of $\tilde{V}_n$ and the fact that $V_0(y) \to 0$ as $|y| \to \infty$, we easily see that Lemma 2.1 implies

$$\big\|\Psi_n(\cdot)/\|\Psi_n\|_\infty - V_0\big(\sigma_n^{1/2}(\cdot - x_n)\big)\big\|_{L^\infty([0,1])} \to 0 \text{ as } n \to \infty. \tag{2.4}$$

We now denote $\tilde{\Psi}_n(x) = \Psi_n(x)/\|\Psi_n\|_\infty$ and consider the function

$$\tilde{u}_n(x) := \sigma_n^{1/2}\Phi_n(x)\tilde{\Psi}_n(x) = \Big(\sigma_n^{1/2}\frac{\|\phi_n\|_\infty}{\|u_n\|_\infty}\Big)u_n.$$

We will show that for large n, $\tilde{u}_n$ behaves like the δ-function concentrating at x_*. Indeed, we have the following result.

Lemma 2.2. *For any given small $\epsilon > 0$, $|x - x_n| \geq \epsilon$ implies*

$$0 < \tilde{u}_n(x) \leq \sigma_n^{1/2} \exp\big[-\frac{\sigma_n}{4(\delta+1)d_1}\epsilon^2\big] \to 0. \tag{2.5}$$

Moreover, when $x_n \to x_* \in (0,1)$,

$$\lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C_0 := \sqrt{2d_1\delta}\int_{-\infty}^{\infty} e^{-x^2}dx. \tag{2.6}$$

Proof. For any given small $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ small so that, when $|x-x_n| \leq \delta_0$,

$$\exp\Big[-\frac{\sigma_n}{4\delta d_1}(x-x_n)^2\Big] \leq \Phi_n(x) \leq \exp\Big[-\frac{\sigma_n(1-\epsilon)}{4\delta d_1}(x-x_n)^2\Big].$$

For any $x \in [0,1]$, we have

$$\exp\Big[-\frac{\sigma_n}{4\delta d_1}(x-x_n)^2\Big] \leq \Phi_n(x) \leq \exp\Big[-\frac{\sigma_n}{4(\delta+1)d_1}(x-x_n)^2\Big].$$

Since $\tilde{\Psi}_n \leq 1$, for $|x - x_n| \geq \epsilon$, we have

$$\tilde{u}_n(x) \leq \sigma_n^{1/2} \exp\big[-\frac{\sigma_n}{4(\delta+1)d_1}\epsilon^2\big] \to 0.$$

This proves (2.5). Moreover, we have

$$\begin{aligned}
\int_0^1 \tilde{u}_n(x)dx &= \int_{x_n-\epsilon}^{x_n+\epsilon} \sigma_n^{1/2}\Phi_n(x)\tilde{\Psi}_n(x)dx + o(1)\\
&= \int_{-\epsilon\sigma_n^{1/2}}^{\epsilon\sigma_n^{1/2}} \Phi_n(x_n + \sigma_n^{-1/2}y)\tilde{V}_n(y)dy + o(1)\\
&= \int_{-\infty}^{\infty} \exp\big[-\frac{y^2}{4d_1\delta}\big]V_0(y)dy + o(1)\\
&= \int_{-\infty}^{\infty} \exp\big[-\frac{y^2}{2d_1\delta}\big]dy + o(1).
\end{aligned}$$

Hence (2.6) holds. For later application, let us also note from the above argument that

$$\lim_{n\to\infty}\int_{x_n}^1 \tilde{u}_n(x)dx = \lim_{n\to\infty}\int_0^{x_n}\tilde{u}_n(x)dx = C_0/2. \tag{2.7}$$

□

Denote $\tau_n := \|\Psi_n\|_\infty \sigma_n^{-1/2}$. We find that

$$u_n(x) = \tau_n \tilde{u}_n(x).$$

Lemma 2.3. *Suppose that $x_n \to x_* \in (0,1)$. Then $\{\tau_n\}$ has a subsequence, still denoted by itself, such that $\tau_n \to \tau_* > 0$. Moreover, τ_* and x_* must satisfy*

$$w_0 e^{-A_0x_* - A\tau_*(C_0/2)} = \alpha\big[v_0 - \frac{\tau_*}{d_2}mC_0(1+\beta^{-1}-x_*)\big], \tag{2.8}$$

and

$$m = \int_0^1 f\Big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\Big) dy. \tag{2.9}$$

Furthermore, by possibly passing to a further subsequence, $u_n \to 0$ in $C([0,1] \setminus [x_* - \epsilon, x_* + \epsilon])$, $\forall \epsilon > 0$, and

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - \max\{x, x_*\}) \tag{2.10}$$

uniformly in $[0,1]$.

Proof. By passing to a subsequence, we have two possible cases:

$$\text{(i)} \quad \tau_n \to \infty, \qquad \text{(ii)} \quad \tau_n \to \tau_* \in [0, \infty).$$

Step 1. Case (i) cannot happen.

Suppose $\tau_n \to \infty$; we are going to derive a contradiction. Denote

$$f_n = f(\min\{\alpha v_n, w_n\}).$$

Since

$$w_n(x_n) \le w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s) ds},$$

and by (2.7)

$$\int_0^{x_n} \tilde{u}_n(s) ds \to C_0/2 > 0,$$

we easily see that $w_n(x_n) \to 0$. It follows that

$$\|f_n\|_\infty = f_n(x_n) = f(w_n(x_n)) \to 0.$$

This implies that

$$\int_0^1 f_n \tilde{u}_n dx \to 0.$$

On the other hand, we may integrate the equation for u_n to obtain

$$\int_0^1 [f_n(x) - m] u_n dx = 0,$$

which implies that

$$\int_0^1 [f_n(x) - m] \tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using (2.6), we obtain

$$m C_0 = \lim_{n \to \infty} \int_0^1 f_n \tilde{u}_n dx = 0,$$

which contradicts our assumption that $m > 0$. Therefore case (i) cannot happen.

Step 2. The limiting profile of u_n and v_n.

We next consider case (ii), namely $\tau_n \to \tau_* \in [0,\infty)$. In this case, due to (2.5), $u_n = \tau_n \tilde{u}_n \to 0$ in $C([0,1] \setminus [x_* - \epsilon, x_* + \epsilon])$, $\forall \epsilon > 0$, and hence

$$\tau_n f_n \tilde{u}_n \to 0 \text{ uniformly in } [0, x_* - \epsilon] \cup [x_* + \epsilon, 1],\ \forall \epsilon > 0. \tag{2.11}$$

Let $\zeta_n = v_0 - v_n$. Then

$$-d_2 \zeta_n'' = \tau_n f_n \tilde{u}_n \text{ in } (0,1),\ \zeta_n'(0) = 0,\ \zeta_n'(1) + \beta \zeta_n(1) = 0. \tag{2.12}$$

Since $v_n \geq 0$ we have $\zeta_n \leq v_0$. Since $\tau_n f_n \tilde{u}_n > 0$, from (2.12) and the maximum principle, we deduce that $\zeta_n > 0$. Hence we always have $0 < \zeta_n \leq v_0$. Therefore we can integrate (2.12) to obtain

$$\eta_n := \tau_n \int_0^1 f_n \tilde{u}_n dx = d_2[\zeta_n'(0) - \zeta_n'(1)] = d_2 \beta \zeta_n(1) \in [0, d_2 \beta v_0].$$

This implies that, by passing to a subsequence, we may assume that $\eta_n \to \eta_* \in [0, d_2 \beta v_0]$.

Moreover, using (2.11), (2.12) and $\eta_n \to \eta_*$, we find that

$$\{\zeta_n'\} \text{ is a bounded sequence in } L^\infty([0,1]),$$
$$\zeta_n'(x) \to 0 \text{ uniformly in } [0, x_* - \epsilon],\ \forall \epsilon > 0,$$
$$\zeta_n'(x) \to -\eta_*/d_2 \text{ uniformly in } [x_* + \epsilon, 1],\ \forall \epsilon > 0.$$

Since moreover $0 \leq \zeta_n \leq v_0$, we conclude that $\{\zeta_n\}$ is precompact in $C([0,1])$. Hence, by passing to a subsequence, we may assume that $\zeta_n \to \zeta$ in $C([0,1])$.

On the other hand, we may apply the L^p theory to (2.12) and the Sobolev imbedding theorem to find a further subsequence, still denoted by ζ_n, such that $\zeta_n \to \tilde{\zeta}$ in $C^1(J)$ for any compact interval $J \subset [0, x_*) \cup (x_*, 1]$, and $\tilde{\zeta}$ satisfies (in the weak sense)

$$-d_2 \tilde{\zeta}'' = 0 \text{ in } [0, x_*) \cup (x_*, 1],\ \tilde{\zeta}'(0) = 0,\ \tilde{\zeta}'(1) + \beta \tilde{\zeta}(1) = 0.$$

Clearly we must have $\tilde{\zeta} = \zeta$. Moreover, our earlier analysis on ζ_n implies that $\zeta'(x) = 0$ in $[0, x_*)$ and $\zeta'(x) = -\eta_*/d_2$ in $(x_*, 1]$. These properties uniquely determine ζ:

$$\zeta(x) = (\eta_*/d_2)(1 + \beta^{-1} - \max\{x_*, x\}). \tag{2.13}$$

Step 3. $\tau_* > 0$.

Otherwise, $\tau_* = 0$ and hence $\eta_* = 0$. It follows that $\zeta = 0$ and $v_n \to v_0$ uniformly in $[0,1]$, and

$$w_n(x) = w_0 e^{-A_0 x} e^{-A \tau_n \int_0^x \tilde{u}_n(s) ds} \to w_0 e^{-A_0 x} = w_*(x)$$

uniformly in $[0,1]$. This implies that

$$x_* = x_0^* \text{ and } f_n(x) \to f_0(x) := f(\min\{\alpha v_0, w_*\}) \text{ uniformly in } [0,1].$$

We may now integrate the equation for u_n to obtain, as before,

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$, we deduce

$$[f_0(x_0^*) - m]C_0 = 0,$$

which contradicts our assumption that $m < f(\min\{\alpha v_0, w_0\}) = f_0(x_0^*)$. Hence $\tau_* > 0$.

Step 4. The equations for x_* and τ_*.

We now set to find the equations that determine x_* and τ_*. By (2.7),

$$w_n(x_n) = w_0 e^{-A_0 x_n} e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds} \to w_0 e^{-A_0 x_*} e^{-A\tau_*(C_0/2)}.$$

On the other hand,

$$w_n(x_n) = \alpha v_n(x_n) \to \alpha\big[v_0 - \zeta(x_*)\big].$$

Thus we necessarily have

$$w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha\big[v_0 - \zeta(x_*)\big] = \alpha\big[v_0 - (\eta_*/d_2)(1 + \beta^{-1} - x_*)\big]. \tag{2.14}$$

Moreover, using (2.5), (2.7) and the fact that $\alpha v_n \to \alpha(v_0 - \zeta)$ uniformly in $[0,1]$, we deduce

$$\int_0^{x_n} f(\alpha v_n)\tilde{u}_n dx \to (C_0/2) f\big(\alpha v_0 - \alpha\zeta(x_*)\big). \tag{2.15}$$

Using

$$w_n(x) = w_0 e^{-A_0 x} e^{-A\tau_n \int_0^x \tilde{u}_n(s)ds}$$

and the property of $\tilde{u}_n$ we obtain, for any small $\epsilon > 0$,

$$\begin{aligned}
\int_{x_n}^1 f(w_n)\tilde{u}_n dx &= \int_{x_n}^1 f\Big(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx \\
&= \int_{x_n}^{x_*+\epsilon} f\Big(w_0 e^{-A_0 x - A\tau_n(C_0/2)} e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx + o(1) \\
&= [1 + o_\epsilon(1)] \int_{x_n}^{x_*+\epsilon} f\Big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx + o(1) \\
&= [1 + o_\epsilon(1)] \int_0^{[\int_{x_n}^1 \tilde{u}_n(s)ds]} f\Big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_n y}\Big) dy + o(1) \\
&= [1 + o_\epsilon(1)] \int_0^{C_0/2} f\Big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\Big) dy + o(1),
\end{aligned}$$

That is,

$$\int_{x_n}^1 f(w_n)\tilde{u}_n(x)dx \to \int_0^{C_0/2} f\Big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\Big) dy \tag{2.16}$$

as $n \to \infty$.

Combining (2.15) and (2.16), we obtain

$$\begin{aligned} \eta_* &= \lim_{n\to\infty} \tau_n \int_0^1 f_n \tilde{u}_n dx \\ &= \tau_* \big[(C_0/2) f\big(\alpha v_0 - \alpha\zeta(x_*)\big) + \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy\big]. \end{aligned} \tag{2.17}$$

Moreover, we may integrate the equation for u_n to obtain

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using (2.15), (2.16), we obtain

$$mC_0 = (C_0/2) f(\alpha v_0 - \alpha\zeta(x_*)) + \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy.$$

This combined with (2.17) yields

$$\eta_* = \tau_* m C_0, \tag{2.18}$$

and combined with (2.14) gives

$$\begin{aligned} m &= (1/2) f(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)}) + C_0^{-1} \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy \\ &= C_0^{-1} \int_0^{C_0} f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, y\}}\big) dy \\ &= \int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big) dy, \end{aligned}$$

so (2.9) is proved. (2.8) and (2.10) clearly follow from (2.13), (2.14) and (2.18). □

We now consider the case $x_* = 0$. By passing to a subsequence, we have two subcases:

$$\text{(a1) } a_n := -\sigma_n^{1/2} x_n \to -\infty, \quad \text{(a2) } a_n \to a_* \in (-\infty, 0].$$

Lemma 2.4. *In subcase* (a1)*, all the conclusions in Lemmas 2.2 and 2.3 hold. In subcase* (a2)*,* $\{\tau_n\}$ *has a subsequence, still denoted by itself, such that* $\tau_n \to \tau_* > 0$*. Moreover,* τ_* *and* a_* *must satisfy*

$$m = \int_0^1 f\big(w_0 e^{-A\tau_* \max\{C(a_*), [C_0/2 + C(a_*)]y\}}\big) dy, \tag{2.19}$$

and

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} m[C_0/2 + C(a_*)](1+\beta^{-1})\Big) = w_0 e^{-A\tau_* C(a_*)} \quad \text{if } a_* < 0, \tag{2.20}$$

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} m(C_0/2)(1+\beta^{-1})\Big) \geq w_0 \quad \text{if } a_* = 0, \tag{2.21}$$

where

$$C(a_*) := \int_{a_*}^{0} \exp\big[-\frac{y^2}{2d_1\delta}\big]dy.$$

Furthermore, by possibly passing to a further subsequence, $u_n \to 0$ in $C([\epsilon, 1])$, $\forall \epsilon \in (0, 1)$,

$$\lim_{n\to\infty} \int_0^{x_n} \tilde{u}_n(x)dx = C(a_*),\ \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C_0/2 + C(a_*), \tag{2.22}$$

and

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m[C_0/2 + C(a_*)](1 + \beta^{-1} - x) \tag{2.23}$$

uniformly in $[0, 1]$.

Proof. In subcase (a1), we may repeat the arguments used for the case $x_* \in (0, 1)$ above to see that all the conclusions there (with x_* replaced by 0) remain valid; the proofs carry over with minor modifications.

Consider now subcase (a2). In this case, we may use interior and boundary L^p estimates and the Sobolev imbedding theorem to conclude that, by passing to a subsequence, $\|\tilde{V}_n - \tilde{V}\|_{C^1([a_n,M])} \to 0$ for all $M > 0$, where $\tilde{V}$ satisfies, instead of (2.3),

$$\begin{cases} -d_1\tilde{V}'' = \dfrac{2d_1\delta - y^2}{4d_1\delta^2}\tilde{V},\ 0 < \tilde{V} \le 1 \text{ in } (a_*, \infty), \\ d_1\tilde{V}'(a_*) + \dfrac{a_*}{2\delta}\tilde{V}(a_*) = 0,\ \tilde{V}(y^*) = 1,\ \tilde{V}'(y^*) = 0. \end{cases} \tag{2.24}$$

Note that as before $\tilde{V}$ is decreasing in $[(2d_1\delta)^{1/2}, \infty)$. This and (2.24) imply that $\tilde{V}$ converges to 0 as $y \to \infty$. Moreover, an elementary consideration shows that

$$|\tilde{V}'(y)|, \tilde{V}(y) \le C_1 e^{-C_2 y}$$

for some $C_1, C_2 > 0$ and all $y > 0$.

We will show that $y^* = 0$ and $\tilde{V}$ is again the unique solution of (2.3) with $y^* = 0$, namely V_0. Since V_0 and $|V_0'|$ are bounded from above by a function of the form $C_1 e^{-C_2|y|}$, we can multiply the first equation in (2.24) by V_0, integrate over $[y^*, \infty)$ and use integration by parts to deduce

$$d_1[\tilde{V}V_0' - \tilde{V}'V_0]\big|_{y^*}^{\infty} = 0.$$

It follows that $V_0'(y^*) = 0$, which implies that $y^* = 0$. Therefore, by the uniqueness of initial value problems of the ordinary differential equations, we deduce $\tilde{V} \equiv V_0$. Let us note that a direct calculation shows

$$d_1V_0'(y) + \frac{y}{2\delta}V_0(y) = 0 \text{ for every } y \in (-\infty, \infty).$$

Therefore (2.24) does not introduce any restriction for a_*.

Since now $\sigma_n^{1/2} x_n \to -a_*$, instead of (2.6), we have

$$\lim_{n\to\infty} \int_0^{x_n} \tilde{u}_n(x)dx = C(a_*), \ \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C_0/2 + C(a_*), \tag{2.25}$$

where

$$C(a_*) := \int_{a_*}^0 \exp\big[-\frac{y^2}{4d_1\delta}\big] V_0(y)dy = \int_{a_*}^0 \exp\big[-\frac{y^2}{2d_1\delta}\big]dy.$$

We proceed as in the case $x_* \in (0,1)$ and have two possibilities for τ_n as before. We show that in the current case, we still cannot have $\tau_n \to \infty$. Arguing indirectly, we assume that $\tau_n \to \infty$.

Then in the case $a_* < 0$, we have $C(a_*) > 0$ and hence

$$w_n(x_n) \le w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds} \to 0.$$

It follows that

$$\|f_n\|_\infty = f_n(x_n) = f(w_n(x_n)) \to 0,$$

and

$$\int_0^1 f_n \tilde{u}_n dx \to 0.$$

If $a_* = 0$, then $C(a_*) = 0$ and

$$\begin{aligned} \int_0^1 f_n(x)\tilde{u}_n(x)dx &= \int_{x_n}^1 f(w_n(x))\tilde{u}_n(x)dx + o(1) \\ &\le \int_{x_n}^1 f\Big(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx + o(1) \\ &= \int_0^{[\int_{x_n}^1 \tilde{u}_n(s)ds]} f\Big(w_0 e^{-A\tau_n y}\Big)dy + o(1) \\ &\le \epsilon f(w_0) + \int_\epsilon^{C_0/2} f\Big(w_0 e^{-A\tau_n y}\Big)dy + o(1) \\ &= \epsilon f(w_0) + o(1), \ \forall \epsilon \in (0, C_0/2). \end{aligned}$$

Therefore we always have

$$\int_0^1 f_n \tilde{u}_n dx \to 0 \text{ as } n \to \infty.$$

As before, we may integrate the equation for u_n to obtain

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using the above estimate we deduce

$$-m[C_0/2 + C(a_*)] = 0,$$

a contradiction to our assumption that $m > 0$. Therefore we cannot have $\tau_n \to \infty$.

Thus we can only have the case $\tau_n \to \tau_*$. Then much as before we deduce $u_n \to 0$ in $C([\epsilon, 1])$ for all $\epsilon \in (0, 1)$, and

$$\zeta_n \to \zeta := (\eta_*/d_2)(1 + \beta^{-1} - x)$$

in $C([0, 1]) \cap C^1([\epsilon, 1])$, $\forall \epsilon \in (0, 1)$. If $\tau_* = 0$, we can deduce as before that $m = f_0(x_0^*)$, a contradiction to our initial assumption on m. Therefore $\tau_* > 0$.

If $a_* = 0$, we first choose $y_n \in (x_n, 1)$ such that $y_n \to 0$ and $\int_{y_n}^1 \tilde{u}_n(x)dx \to 0$, and then we have

$$\begin{aligned}
\int_0^1 f_n(x)\tilde{u}_n(x)dx &= \int_{x_n}^{y_n} f(w_n(x))\tilde{u}_n(x)dx + o(1) \\
&= \int_{x_n}^{y_n} f\left(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\right)\tilde{u}_n dx + o(1) \\
&= \int_{x_n}^{y_n} f\left(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\right)\tilde{u}_n dx + o(1) \\
&= \int_{x_n}^{1} f\left(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\right)\tilde{u}_n dx + o(1) \\
&= \int_0^{C_0/2} f\left(w_0 e^{-A\tau_* y}\right)dy + o(1).
\end{aligned}$$

If $a_* < 0$, then $x_n > 0$ and $w_n(x_n) = \alpha v_n(x_n)$. From

$$v_n(x_n) \to v_0 - \zeta(0)$$

and

$$w_n(x_n) = w_0 e^{-A_0 x_n - A\tau_n \int_0^{x_n} \tilde{u}_n dx} \to w_0 e^{-A\tau_* C(a_*)}$$

we obtain

$$\alpha[v_0 - \zeta(0)] = w_0 e^{-A\tau_* C(a_*)}.$$

Moreover, similar to the above,

$$\begin{aligned}
\int_{x_n}^1 f_n(x)\tilde{u}_n(x) &= \int_{x_n}^{y_n} f(w_n(x))\tilde{u}_n(x)dx + o(1) \\
&= \int_{x_n}^{y_n} f\left(w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\right)\tilde{u}_n dx + o(1) \\
&= \int_0^{C_0/2} f\left(w_0 e^{-A\tau_* C(a_*) - A\tau_* y}\right)dy + o(1),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{x_n} f_n(x)\tilde{u}_n(x)dx &= \int_0^{x_n} f(\alpha v_n(x))\tilde{u}_n(x)dx \\
&= f\big(\alpha[v_0 - \zeta(0)]\big)C(a_*) + o(1) \\
&= C(a_*) f\left(w_0 e^{-A\tau_* C(a_*)}\right) + o(1).
\end{aligned}$$

Therefore we always have

$$\int_0^1 f_n\tilde{u}_n dx \to \int_0^{C_0/2+C(a_*)} f\Big(w_0 e^{-A\tau_* \max\{C(a_*),y\}}\Big)dy. \tag{2.26}$$

We may now use

$$\int_0^1 [f_n(x)-m]\tilde{u}_n dx = 0$$

to obtain

$$m[C_0/2+C(a_*)] = \int_0^{C_0/2+C(a_*)} f\Big(w_0 e^{-A\tau_* \max\{C(a_*),y\}}\Big)dy.$$

Therefore

$$m = \int_0^1 f\Big(w_0 e^{-A\tau_* \max\{C(a_*),[C_0/2+C(a_*)]y\}}\Big)dy,$$

and (2.19) is proved.

We thus obtain

$$\eta_* = \tau_* \lim_{n\to\infty}\int_0^1 f_n\tilde{u}_n dx = \tau_* m[C_0/2+C(a_*)].$$

Therefore,

$$v_n(x) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2} m[C_0/2+C(a_*)](1+\beta^{-1}-x)$$

uniformly in $[0,1]$, that is, (2.23) holds.

Let us note that (2.22) was already proved in (2.25). So it remains to prove (2.20) and (2.21). If $a_*<0$, then $x_n>0$ and we necessarily have $\alpha v_n(x_n) = w_n(x_n)$. Recall that

$$w_n(x_n) \to w_0 e^{-A\tau_* C(a_*)},\ v_n(x_n) \to v_0 - \zeta(0).$$

Hence

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} m[C_0/2+C(a_*)](1+\beta^{-1})\Big) = w_0 e^{-A\tau_* C(a_*)}.$$

If $a_*=0$, then $x_n=0$ is possible and so we have $\alpha v_n(x_n)\ge w_n(x_n)$ in general, and instead of the above identity we should have

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} m(C_0/2)(1+\beta^{-1})\Big) \ge w_0.$$

Thus (2.20) and (2.21) are established. The proof is now complete. □

Finally we consider the case $x_*=1$. By passing to a subsequence, we have two subcases:

$$\text{(b1) } b_n := \sigma_n^{1/2}(1-x_n)\to\infty,\ \ \text{(b2) } b_n\to b_*\in[0,\infty).$$

Lemma 2.5. *In subcase* (b1)*, all the conclusions in Lemmas 2.2 and 2.3 holds. In subcase* (b2)*,* $\{\tau_n\}$ *has a subsequence, still denoted by itself, such that* $\tau_n \to \tau_* > 0$*. Moreover,* τ_* *and* b_* *must satisfy*

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, [C_0/2+\tilde{C}(b_*)]y\}}\big)dy \tag{2.27}$$

and

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}[C_0/2 + \tilde{C}(b_*)]\Big) = w_0 e^{-A_0 - A\tau_* C_0/2} \quad \textit{if } b_* > 0, \tag{2.28}$$

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}(C_0/2)\Big) \le w_0 e^{-A_0 - A\tau_* C_0/2} \quad \textit{if } b_* = 0, \tag{2.29}$$

where

$$\tilde{C}(b_*) := \int_0^{b_*} \exp\big[-\frac{y^2}{2d_1\delta}\big]dy = C(-b_*).$$

Furthermore, by possibly passing to a further subsequence, $u_n \to 0$ *in* $C([0, 1-\epsilon])$ *for every* $\epsilon \in (0,1)$*,*

$$\lim_{n\to\infty} \int_{x_n}^1 \tilde{u}_n(x)dx = \tilde{C}(b_*), \ \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C_0/2 + \tilde{C}(b_*), \tag{2.30}$$

$$v_n(x) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2\beta}[C_0/2 + \tilde{C}(b_*)] \tag{2.31}$$

uniformly in $[0,1]$*.*

Proof. In subcase (b1), we may repeat the arguments used in Lemmas 2.2 and 2.3 for the case $x_* \in (0,1)$ to see that all the conclusions there (with x_* replaced by 1) remain valid; the proofs only need minor modifications.

We now consider subcase (b2). Then instead of (2.3) we have

$$\begin{cases} -d_1\tilde{V}'' = \dfrac{2d_1\delta - y^2}{4d_1\delta}\tilde{V}, \ 0 < \tilde{V} \le 1 \text{ in } (-\infty, b_*), \\ \tilde{V}'(b_*) + \dfrac{b_*}{2\delta}\tilde{V}(b_*) = 0, \ \tilde{V}(y^*) = 1, \ \tilde{V}'(y^*) = 0. \end{cases} \tag{2.32}$$

Note that as before $\tilde{V}$ is increasing in $(-\infty, -(2d_1\delta)^{1/2}]$. This and (2.32) imply that $\tilde{V}$ converges to 0 as $y \to -\infty$. Moreover, an elementary consideration shows that

$$|\tilde{V}'(y)|, \tilde{V}(y) \le C_1 e^{-C_2|y|}$$

for some $C_1, C_2 > 0$ and all $y < 0$.

As in the case for (2.24), we can similarly show that $y^* = 0$ and $\tilde{V} \equiv V_0$, the unique solution of (2.3) with $y^* = 0$. Moreover, (2.32) introduces no restriction for b_*.

Since $\sigma_n^{1/2}(1-x_n)\to b_*$, instead of (2.6), we have

$$\lim_{n\to\infty}\int_{x_n}^1 \tilde{u}_n(x)dx=\tilde{C}(b_*),\ \lim_{n\to\infty}\int_0^1 \tilde{u}_n(x)dx=C_0/2+\tilde{C}(b_*),$$

where

$$\tilde{C}(b_*):=\int_0^{b_*}\exp\big[-\frac{y^2}{4d_1\delta}\big]V_0(y)dy=C(-b_*).$$

This establishes (2.30).

We proceed as in the case $x_*\in(0,1)$ and have two possibilities for τ_n as before. We show that in the current case, we still cannot have $\tau_n\to\infty$. Arguing indirectly, we assume that $\tau_n\to\infty$.

Since $\int_0^{x_n}\tilde{u}_n dx\to C_0/2$, we have

$$w_n(x_n)\le w_0 e^{-A\tau_n\int_0^{x_n}\tilde{u}_n(s)ds}\to 0.$$

It follows that

$$\|f_n\|_\infty=f_n(x_n)\le f(w_n(x_n))\to 0,$$

and

$$\int_0^1 f_n\tilde{u}_n dx\to 0.$$

As before, we may integrate the equation for u_n to obtain

$$\int_0^1[f_n(x)-m]\tilde{u}_n dx=0.$$

Letting $n\to\infty$ and using the above estimate we deduce

$$-m[C_0/2+\tilde{C}(b_*)]=0,$$

a contradiction to our assumption that $m>0$. Therefore we cannot have $\tau_n\to\infty$.

Thus we can only have the case $\tau_n\to\tau_*$. Then much as before we deduce $u_n\to 0$ in $C([0,1-\epsilon])$ for each $\epsilon\in(0,1)$ and $\zeta_n\to\zeta$ in $C([0,1])\cap C^1([0,1-\epsilon])$, $\forall\epsilon\in(0,1)$, with ζ satisfying

$$\zeta''=0 \text{ in } [0,1),\ \zeta'=0 \text{ in } [0,1).$$

Hence ζ is a constant. To determine its value, we use

$$-d_2\zeta_n'(1)=\int_0^1\tau_n f_n\tilde{u}_n dx\to\tau_*[C_0/2+\tilde{C}(b_*)]$$

and

$$\zeta_n'(1)+\beta\zeta_n(1)=0$$

to deduce

$$-\frac{\tau_*}{d_2}[C_0/2+\tilde{C}(b_*)]+\beta\zeta=0,$$

and hence

$$\zeta = \frac{\tau_*}{d_2\beta}[C_0/2 + \tilde{C}(b_*)]. \tag{2.33}$$

If $\tau_* = 0$, then $\zeta \equiv 0$ and hence $v_n \to v_0$ uniformly in $[0, 1]$ and

$$w_n(x) = w_0 e^{-A_0 x - A\tau_n \int_0^x \tilde{u}_n dx} \to w_0 e^{-A_0 x}$$

uniformly in $[0, 1]$. Then we can deduce as before that $m = f_0(x_0^*)$, a contradiction to our initial assumption on m. Therefore $\tau_* > 0$.

We have

$$\begin{aligned} \int_0^{x_n} f_n(x)\tilde{u}_n(x)dx &= \int_0^{x_n} f(\alpha v_n(x))\tilde{u}_n(x)dx \\ &= (C_0/2) f\big(\alpha(v_0 - \zeta)\big) + o(1). \end{aligned}$$

If $b_* = 0$, then

$$\int_{x_n}^1 f_n(x)\tilde{u}_n(x)dx = o(1).$$

If $b_* > 0$, then $x_n > 0$ and $w_n(x_n) = \alpha v_n(x_n)$. From

$$v_n(x_n) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2\beta}[C_0/2 + \tilde{C}(b_*)]$$

and

$$w_n(x_n) = w_0 e^{-A_0 x_n - A\tau_n \int_0^{x_n} \tilde{u}_n dx} \to w_0 e^{-A_0 - A\tau_* C_0/2}$$

we obtain

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}[C_0/2 + \tilde{C}(b_*)]\Big) = w_0 e^{-A_0 - A\tau_* C_0/2}. \tag{2.34}$$

Moreover,

$$\begin{aligned} \int_{x_n}^1 f_n(x)\tilde{u}_n(x)dx &= \int_{x_n}^1 f(w_n(x))\tilde{u}_n(x)dx \\ &= \int_{x_n}^1 f\Big(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx \\ &= \int_0^{\tilde{C}(b_*)} f\Big(w_0 e^{-A_0 - A\tau_* C_0/2 - A\tau_* y}\Big)dy + o(1). \end{aligned}$$

Therefore, whether $b_* = 0$ or $b_* > 0$, we always have

$$\int_0^1 f_n(x)\tilde{u}_n(x)dx \to \int_0^{C_0/2 + \tilde{C}(b_*)} f\Big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, y\}}\Big)dy. \tag{2.35}$$

We may now use

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0,$$

to obtain

$$m[C_0/2+\tilde{C}(b_*)] = \int_0^{C_0/2+\tilde{C}(b_*)} f\big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, y\}}\big)dy,$$

which gives (2.27).

Note that if $b_* = 0$, then $x_n = 1$ is possible and we only have $w_n(x_n) \geq \alpha v(x_n)$, so instead of (2.27), we should have

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}[C_0/2+\tilde{C}(b_*)]\Big) \leq w_0 e^{-A_0 - A\tau_* C_0/2}.$$

Thus we have established (2.28) and (2.29). Clearly (2.31) follows from (2.33) and the fact that $v_n \to v_0 - \zeta$ uniformly in $[0,1]$. The proof is complete. □

3. Limiting profile of the positive solutions

We are now ready to state and prove our main results. We will show that the limiting equations obtained in the previous section uniquely determine x_* and τ_*, and the value of v_0 determines which set of limiting equations should be used for calculating x_* and τ_*. In this way, the asymptotic behavior of the positive solutions is completely determined.

Let us recall that m is fixed such that

$$0 < m < f(\min\{\alpha v_0, w_0\}), \tag{3.1}$$

and $\sigma_n \to \infty$ is a sequence of positive numbers. Therefore by Theorems 1.1 and 1.2, problem (1.1) with $\sigma = \sigma_n$ has a positive solution (u_n, v_n) for all large n. Recall that $C_0 > 0$ is given in (2.6), which is completely determined by δ and d_1. Due to (3.1) there exists a unique $\tau_0^* > 0$ such that

$$m = \int_0^1 f\big(w_0 e^{-A\tau_0^* \max\{C_0/2, C_0 y\}}\big)dy. \tag{3.2}$$

Let us then define

$$v^*(m) := \frac{w_0}{\alpha} e^{-A\tau_0^* C_0/2} + \frac{\tau_0^*}{d_2} m C_0 (1+\beta^{-1}). \tag{3.3}$$

Let $\underline{v}(m) > 0$ be uniquely determined by

$$m = f(\alpha \underline{v}(m)).$$

By (3.1), we always have $v_0 > \underline{v}(m)$.

When $m < f(w_0 e^{-A_0})$, we can find a unique $\tau_1^* > 0$ such that

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_1^* \max\{C_0/2, C_0 y\}}\big)dy. \tag{3.4}$$

We now define

$$v_*(m) := \begin{cases} \dfrac{w_0}{\alpha} e^{-A_0 - A\tau_1^* C_0/2} + \dfrac{\tau_1^*}{d_2} m C_0 \beta^{-1} & \text{if } m < f(w_0 e^{-A_0}), \\ \underline{v}(m) & \text{if } f(w_0 e^{-A_0}) \le m < f(w_0). \end{cases} \tag{3.5}$$

It is easily seen that $v_*(m)$ is continuous.

As we will see below, to completely determine the asymptotic profile of (u_n, v_n), it is necessary to distinguish the cases $v_0 \in [v_*(m), v^*(m)]$, $v_0 > v^*(m)$, and $v_0 < v_*(m)$.

Theorem 3.1. *Suppose that $v_0 > \underline{v}(m)$ and*

$$v_*(m) \le v_0 \le v^*(m). \tag{3.6}$$

Then the system (2.8) and (2.9), namely

$$\begin{cases} w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha\big[v_0 - \dfrac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - x_*)\big], \\ m = \displaystyle\int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big) dy, \end{cases}$$

has a unique solution pair (x_*, τ_*) satisfying $x_* \in [0,1]$ and $\tau_* > 0$. Moreover,

$$u_n \to 0 \text{ in } C([0,1] \setminus [x_* - \epsilon, x_* + \epsilon]),\ \forall \epsilon > 0,\quad \int_0^1 u_n dx \to \tau_* C_0,$$

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - \max\{x, x_*\}) \text{ uniformly in } [0,1].$$

Furthermore, $x_* = 0$ if $v_0 = v^*(m)$, $x_* \in (0,1)$ if $v_*(m) < v_0 < v^*(m)$, $x_* = 1$ if $v_0 = v_*(m)$.

Proof. Using the notations of the previous section, by passing to a subsequence, $x_n \to x_* \in [0,1]$. By possibly passing to a further subsequence, the behavior of (u_n, v_n) as $n \to \infty$ is then determined by Lemmas 2.2, 2.3, 2.4 and 2.5, according to whether $x_* \in (0,1)$, $x_* = 0$ or $x_* = 1$. In the last two cases we have to further distinguish whether it is subcase (a1) or (a2), and subcase (b1) or (b2), respectively.

If we can show that x_* and τ_* are uniquely determined by the value of v_0, then the corresponding results in the previous section would hold not only for a subsequence, but for the entire original sequence, and hence the behavior of (u_n, v_n) as $n \to \infty$ would be completely determined.

The rather long proof below is broken into several steps.

Step 1. Subcases (a2) and (b2) do not happen.

Firstly we observe that subcase (a2) does not happen. Indeed, if this case happens, then since $C(a_*) < C_0/2$, as explained below, we see from a careful comparison of (2.19) and (3.2) that

$$\tau_* > \tau_0^*,\ \tau_* C(a_*) < \tau_0^* C_0/2,\ \tau_*[C_0/2 + C(a_*)] > \tau_0^* C_0.$$

In the comparison, we can deduce these inequalities one at a time, in the above order, and the previous inequalities are used for obtaining the next inequality. For example, to deduce $\tau_* C(a_*) < \tau_0^* C_0/2$ from $\tau_* > \tau_0^*$, we observe that $\tau_* C(a_*) \geq \tau_0^* C_0/2$ and $\tau_* > \tau_0^*$ would imply

$$\begin{aligned}\tau_* \max\{C(a_*), [C_0/2 + C(a_*)]y\} &\geq \max\{\tau_0^* C_0/2, [\tau_* C_0/2 + \tau_0^* C_0/2]y\} \\ &\geq \tau_0^* \max\{C_0/2, C_0 y\},\end{aligned}$$

with strict inequality holding in the last step for $y \in [1/2, 1]$, which is impossible when one compares (2.19) with (3.2).

It then follows from (2.20) and (2.21) that $v_0 > v^*(m)$, contradicting (3.6).

Similarly, if subcase (b2) happens, then from (2.27) we deduce

$$\tau_* > \tau_1^* \text{ and } \tau_*[C_0/2 + \tilde{C}(b_*)] < \tau_1^* C_0,$$

which imply, by (2.28) and (2.29) that $v_0 < v_*(m)$, again contradicting (3.6). Therefore subcase (b2) cannot happen.

Thus, by our discussion in the previous section, we are in the cases where (2.8) and (2.9) hold. To show that (2.8) and (2.9) have a unique solution (x_*, τ_*) satisfying $x_* \in [0, 1]$ and $\tau_* > 0$, we establish below a procedure to uniquely find x_* and τ_*. In the discussion below, we will treat $v_0 > 0$ as a varying parameter.

Step 2. A procedure to solve (2.8) and (2.9).

It is useful to use the new variable

$$\lambda = A_0 x_* + A\tau_* C_0/2.$$

Then

$$x_* = (\lambda - A\tau_* C_0/2)/A_0,$$

and (2.8) can be rewritten as

$$\frac{w_0}{\alpha} e^{-\lambda} = v_0 - \frac{\tau_*}{d_2} m C_0 \Big(1 + \beta^{-1} - \frac{\lambda - A\tau_* C_0/2}{A_0}\Big),$$

or

$$\frac{mC_0}{d_2 A_0} \tau_* \big[(1 + \beta^{-1})A_0 - \lambda + A(C_0/2)\tau_*\big] = v_0 - \frac{w_0}{\alpha} e^{-\lambda}.$$

We now consider the quadratic equation of τ:

$$\frac{mC_0}{d_2 A_0} \tau \big[(1 + \beta^{-1})A_0 - \lambda + A(C_0/2)\tau\big] = v_0 - \frac{w_0}{\alpha} e^{-\lambda}. \tag{3.7}$$

For each $v_0 > 0$, let $\lambda_0(v_0)$ denote the minimal nonnegative λ such that $v_0 - \frac{w_0}{\alpha}e^{-\lambda} \geq 0$. Clearly

$$\lambda_0(v_0) = 0 \text{ if } v_0 \geq w_0/\alpha,\ \lambda_0(v_0) \text{ is decreasing in } (0, w_0/\alpha],\ \lim_{v_0 \to 0} \lambda_0(v_0) = \infty. \tag{3.8}$$

For each $v_0 > 0$ and $\lambda \geq \lambda_0(v_0)$, the quadratic equation (3.7) has a maximal zero, which we denote by $\tau(\lambda, v_0)$. It is easily seen that $\tau(\lambda, v_0) \geq 0$ and

$$\text{when } v_0 \leq w_0/\alpha,\ \tau(\lambda_0(v_0), v_0) = \max\Big\{0, \frac{\lambda_0(v_0) - A_0(1+\beta^{-1})}{AC_0/2}\Big\}, \tag{3.9}$$

$$\tau(\lambda, v_0) \text{ is increasing in } \lambda \text{ and in } v_0,\ \lim_{v_0 \to \infty} \tau(\lambda, v_0) = \infty \text{ for fixed } \lambda \geq 0. \tag{3.10}$$

Since $\lambda_0(w_0/\alpha) = 0$, by (3.9), $\tau(\lambda_0(w_0/\alpha), w_0/\alpha) = 0$. Let us consider the continuous function

$$M(v_0) = \int_0^1 f\big(w_0 e^{-\lambda_0(v_0) - A\tau(\lambda_0(v_0), v_0)\max\{0, C_0 y - C_0/2\}}\big)dy.$$

The above observation shows that $M(w_0/\alpha) = f(w_0) > m$. By (3.8), we have $M(v_0) \to 0$ as $v_0 \to 0$. By (3.10), we deduce $M(v_0) \to 0$ as $v_0 \to \infty$. Hence from the continuity of $M(v_0)$ we can find v_{min} and v_{max} such that

$$0 < v_{min} < w_0/\alpha < v_{max} < \infty,$$

$$M(v_0) > m\ \forall v_0 \in (v_{\min}, v_{max}),\ M(v_{min}) = M(v_{max}) = m.$$

Now for each $v_0 \in (v_{min}, v_{max})$,

$$m < \int_0^1 f\big(w_0 e^{-\lambda_0(v_0) - A\tau(\lambda_0(v_0), v_0)\max\{0, C_0 y - C_0/2\}}\big)dy.$$

This and the monotonicity of $\tau(\lambda, v_0)$ in λ imply that for such v_0 we can find a unique $\lambda_* = \lambda_*(v_0) > \lambda_0(v_0)$ such that

$$m = \int_0^1 f\big(w_0 e^{-\lambda_* - A\tau(\lambda_*, v_0)\max\{0, C_0 y - C_0/2\}}\big)dy.$$

Clearly $v_0 \to \lambda_*(v_0)$ is continuous and

$$\lambda_*(v_{min} + 0) = \lambda_0(v_{min}),\ \lambda_*(v_{max} - 0) = \lambda_0(v_{max}).$$

So we may define

$$\lambda_*(v_{min}) = \lambda_0(v_{min}),\ \lambda_*(v_{max}) = \lambda_0(v_{max}).$$

We claim that the function $T(v_0) := \tau(\lambda_*(v_0), v_0)$ is increasing in $[v_{min}, v_{max}]$. Otherwise, we can find $v_{min} \leq s_1 < s_2 \leq v_{max}$ such that $T(s_1) \geq T(s_2)$. Since

$$\int_0^1 f\big(w_0 e^{-\lambda_*(s_1) - AT(s_1)\max\{0, C_0 y - C_0/2\}}\big) = \int_0^1 f\big(w_0 e^{-\lambda_*(s_2) - AT(s_2)\max\{0, C_0 y - C_0/2\}}\big),$$

$T(s_1) \geq T(s_2)$ implies that $\lambda_*(s_1) \leq \lambda_*(s_2)$, which implies, by the monotonicity of $\tau(\lambda, v_0)$,

$$T(s_1) = \tau(\lambda_*(s_1), s_1) < \tau(\lambda_*(s_2), s_2) = T(s_2).$$

This contradiction proves the claimed monotonicity of $T(v_0)$.

We show next that $T(v_{max}) > \tau_0^*$. Since $v_{max} > w_0/\alpha$, we have $\lambda_0(v_{max}) = 0$ and hence

$$m = M(v_{max}) = \int_0^1 f(w_0 e^{-A\tau(0,v_{max})\max\{0, C_0 y - C_0/2\}})dy.$$

By (3.2),

$$m = \int_0^1 f(w_0 e^{-A\tau_0^* C_0/2 - A\tau_0^* \max\{0, C_0 y - C_0/2\}})dy.$$

Comparing the above two expressions we obtain $\tau(0, v_{max}) > \tau_0^*$. Hence

$$T(v_{max}) = \tau(\lambda_*(v_{max}), v_{max}) = \tau(\lambda_0(v_{max}), v_{max}) = \tau(0, v_{max}) > \tau_0^*,$$

as we wanted.

We now consider $T(v_{min})$. We have two different cases: $m < f(w_0 e^{-A_0})$ and $m \geq f(w_0 e^{-A_0})$. Consider first the case $m < f(w_0 e^{-A_0})$. We show that $T(v_{min}) < \tau_1^*$ in this case. Since $\lambda_*(v_{\min}) = \lambda_0(v_{\min})$ we have

$$T(v_{min}) = \tau(\lambda_0(v_{min}), v_{min}).$$

Hence, by (3.9),

$$T(v_{min}) = \max\Big\{0, \frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2}\Big\}.$$

If $\frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2} \leq 0$, then $T(v_{min}) = 0 < \tau_1^*$. If $\frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2} > 0$, then

$$T(v_{min}) = \frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2},$$

and hence

$$\begin{aligned} m &= \int_0^1 f\big(w_0 e^{-\lambda_0(v_{min}) - AT(v_{min})\max\{0, C_0 y - C_0/2\}}\big)dy \\ &= \int_0^1 f\big(w_0 e^{-A_0(1+\beta^{-1}) - AT(v_{min})C_0/2 - AT(v_{min})\max\{0, C_0 y - C_0/2\}}\big)dy \\ &= \int_0^1 f\big(w_0 e^{-A_0(1+\beta^{-1}) - AT(v_{min})\max\{C_0/2, C_0 y\}}\big)dy. \end{aligned}$$

Comparing this with (3.4), we find that $T(v_{min}) < \tau_1^*$.

With the above properties of $T(v_0)$, we can uniquely determine v_* and v^* with $v_{min} < v_* < v^* < v_{max}$ such that

$$T(v^*) = \tau_0^*,\ T(v_*) = \tau_1^*.$$

We claim that $v^* = v^*(m)$ and $v_* = v_*(m)$. Indeed, from

$$m = \int_0^1 f\big(w_0 e^{-\lambda_*(v^*) - AT(v^*)\max\{0, C_0 y - C_0/2\}}\big) dy$$

and $T(v^*) = \tau_0^*$, we easily see by comparing with (3.2) that $\lambda_*(v^*) = \tau_0^* AC_0/2$. Hence

$$\tau_0^* = T(v^*) = \tau(\lambda_*(v^*), v^*) = \tau(\tau_0^* AC_0/2, v^*).$$

By the definition of $\tau(\lambda, v_0)$, the above identity means that $\tau = \tau_0^*$ solves (3.7) with $\lambda = \tau_0^* AC_0/2$ and $v_0 = v^*$. Therefore we may compare with (3.3) to deduce $v^* = v^*(m)$. Similarly, we can show that $v_* = v_*(m)$.

Since T is monotone, for each $v_0 \in [v_*(m), v^*(m)]$, $T(v_0) \in [\tau_1^*, \tau_0^*]$. Hence we can compare (3.2) and (3.4) with

$$m = \int_0^1 f\big(w_0 e^{-\lambda_*(v_0) - AT(v_0)\max\{0, C_0 y - C_0/2\}}\big) dy$$

to find that, for such v_0, we necessarily have

$$AT(v_0)C_0/2 \le \lambda_*(v_0) \le A_0 + AT(v_0)C_0/2;$$

otherwise we would arrive at contradictions to $T(v_0) \in [\tau_1^*, \tau_0^*]$. This implies that there exists a unique $x_* \in [0, 1]$ such that

$$\lambda_*(v_0) = A_0 x_* + AT(v_0)C_0/2.$$

Let $\tau_* = T(v_0)$; we find that (x_*, τ_*) solves (2.8) and (2.9).

We next consider the case $m \ge f(w_0 e^{-A_0})$. In this case, $v_*(m) = \underline{v}(m)$; moreover, we show that

$$T(v_{min}) = 0, \quad v_{min} = \underline{v}(m).$$

Indeed, from

$$T(v_{min}) = \tau(\lambda_*(v_{min}), v_{min}) = \tau(\lambda_0(v_{min}), v_{min})$$

we obtain

$$\begin{aligned} m &= \int_0^1 f\big(w_0 e^{-\lambda_0(v_{min}) - A\tau(\lambda_0(v_{min}), v_{min})\max\{0, C_0 y - C_0/2\}}\big) dy \\ &\le f(w_0 e^{-\lambda_0(v_{min})}). \end{aligned}$$

It follows that $\lambda_0(v_{min}) \le A_0 < A_0(1 + \beta^{-1})$. By (3.9), we deduce $\tau(\lambda_0(v_{min}), v_{min}) = 0$, that is, $T(v_{min}) = 0$. This gives

$$m = \int_0^1 f(w_0 e^{-\lambda_0(v_{min})}) dy = f(w_0 e^{\lambda_0(v_{min})}).$$

Hence

$$\alpha \underline{v}(m) = w_0 e^{-\lambda_0(v_{min})}.$$

On the other hand, since $v_{min} < w_0/\alpha$, by the definition of the function λ_0,

$$v_{min} - \frac{w_0}{\alpha} e^{-\lambda_0(v_{min})} = 0.$$

Therefore we have $v_{min} = \underline{v}(m)$.

We can now conclude that there exists a unique $v^* \in (v_{min}, v_{max})$, such that $T(v^*) = \tau_0^*$. We may then prove $v^* = v^*(m)$ as before. Since T is monotone, for each $v_0 \in (v_*(m), v^*(m)] = (\underline{v}(m), v^*(m)]$, $T(v_0) \in (0, \tau_0^*]$. Hence we can compare (3.2) and $m \geq f(w_0 e^{-A_0})$ with (3.1) to deduce

$$AT(v_0)C_0/2 \leq \lambda_*(v_0) < A_0 + AT(v_0)C_0/2,$$

and there exists a unique $x_* \in [0, 1)$ such that

$$\lambda_*(v_0) = A_0 x_* + AT(v_0)C_0/2.$$

Let $\tau_* = T(v_0)$; we find that (x_*, τ_*) solves (2.8) and (2.9).

The above discussion shows that when (3.6) holds, (2.8) and (2.9) has at least one solution (x_*, τ_*) satisfying $x_* \in [0, 1]$ and $\tau_* > 0$, and such a solution can be found by following the above given procedure.

Step 3. Uniqueness of (x_*, τ_*) and completion of the proof.

We next show that when (3.6) holds, (2.8) and (2.9) has a unique solution (x_*, τ_*) satisfying $x_* \in [0, 1]$ and $\tau_* > 0$. So let (x_*, τ_*) be an arbitrary solution of (2.8) and (2.9) with $v_0 \in [v_*(m), v^*(m)] \cap (\underline{v}(m), v^*(m)]$ and $x_* \in [0, 1]$, $\tau_* > 0$. Then τ_* must be the maximal zero of (3.7) with $\lambda = A_0 x_* + A\tau_* C_0/2 > 0$; this is the case because $v_0 - \frac{w_0}{\alpha} e^{-\lambda} > 0$ and thus the two zeros of (3.7) are of opposite sign. Therefore, using our earlier notations,

$$\tau_* = \tau(\lambda, v_0), \ \lambda > \lambda_0(v_0).$$

Then (2.9) yields

$$\begin{aligned} m &= \int_0^1 f(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}) dy \\ &= \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda, v_0) \max\{0, C_0 y - C_0/2\}}) dy. \end{aligned}$$

Since $v_0 \in [v_*(m), v^*(m)] \cap (\underline{v}(m), v^*(m)] \subset (v_{min}, v_{max})$, in view of the above identity, our definition of $\lambda_*(v_0)$ implies that $\lambda = \lambda_*(v_0)$ and hence $\tau(\lambda, v_0) = T(v_0)$, that is $\tau_* = \tau(\lambda, v_0) = T(v_0)$. This implies that the solution pair (x_*, τ_*) is the same as the one obtained through our above introduced procedure for solving (2.8) and (2.9). Hence there is a unique solution.

With τ_* and x_* uniquely determined now, it is easily seen that our conclusions for u_n and v_n follow from Lemmas 2.2, 2.3, 2.4 and 2.5.

Moreover, from the above given procedure for finding (x_*, τ_*), we easily see that $x_* = 0$ if $v_0 = v^*(m)$, $x_* \in (0,1)$ if $v_*(m) < v_0 < v^*(m)$, $x_* = 1$ if $v_0 = v_*(m)$.

The proof of the theorem is now complete. □

Next we consider the case that $v_0 > v^*(m)$. Let $0 < \lambda_0 < \lambda^0$ be uniquely determined by

$$m = f(w_0 e^{-A\lambda_0}) = \int_0^1 f\big(w_0 e^{-A\lambda^0 y}\big) dy. \tag{3.11}$$

For each $\lambda \in [0, \lambda_0]$, we can find a unique $\Gamma = \Gamma(\lambda)$ such that

$$m = \int_0^1 f\big(w_0 e^{-A\max\{\lambda, \Gamma y\}}\big) dy. \tag{3.12}$$

Moreover, it is easily seen that $\lambda \to \Gamma(\lambda)$ is a continuous decreasing function with

$$\Gamma(\lambda_0) = \lambda_0,\ \Gamma(0) = \lambda^0.$$

Therefore we can find a unique $\lambda_*^0 \in (0, \lambda_0)$ such that

$$\Gamma(\lambda_*^0) = 2\lambda_*^0.$$

Comparing with (3.2) we find that actually

$$\lambda_*^0 = \tau_0^* C_0/2. \tag{3.13}$$

We define

$$\Lambda(\lambda) := \frac{w_0}{\alpha} e^{-A\lambda} + \frac{\Gamma(\lambda)}{d_2} m(1+\beta^{-1}).$$

Clearly $\Lambda(\lambda)$ is a decreasing function on $[0, \lambda_0]$ with

$$\Lambda(0) = \frac{w_0}{\alpha} + \frac{\lambda^0}{d_2} m(1+\beta^{-1}),\ \Lambda(\lambda_*^0) = \frac{w_0}{\alpha} e^{-A\lambda_*^0} + \frac{2\lambda_*^0}{d_2} m(1+\beta^{-1}).$$

Due to (3.13), we find that

$$\Lambda(\lambda_*^0) = v^*(m).$$

Theorem 3.2. *Suppose that*

$$v_0 > v^*(m) = \Lambda(\lambda_*^0). \tag{3.14}$$

If $v_0 < \Lambda(0)$ and $\lambda^* \in (0, \lambda_*^0)$ is uniquely determined by $v_0 = \Lambda(\lambda^*)$, then

$$u_n \to 0 \text{ in } C([\epsilon, 1]),\ \forall \epsilon \in (0,1),\ \int_0^1 u_n dx \to \Gamma(\lambda^*),$$

$$v_n(x) \to v_0 - \frac{\Gamma(\lambda^*)}{d_2} m(1+\beta^{-1} - x) \text{ uniformly in } [0,1].$$

If $v_0 \geq \Lambda(0)$, then the above conclusions hold with $\lambda^* = 0$.

Proof. We first show that case (a2) happens. Let us start by observing that none of the cases leading to (2.8) and (2.9) can happen. Indeed, in these cases, (x_*, τ_*) solves (2.8) and (2.9) with $x_* \in [0,1]$ and $\tau_* > 0$. As in Step 3 of the proof of Theorem 3.1, denoting $\lambda = A_0 x_* + A\tau_* C_0/2$, we must have $\tau_* = \tau(\lambda, v_0)$ and $\lambda > \lambda_0(v_0)$. Then (2.9) gives

$$m = \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda, v_0)\max\{0, C_0 y - C_0/2\}})dy.$$

Since $v_0 > v^*(m)$, we have either

$$v_0 > v_{max} \text{ or } v_0 \in (v^*(m), v_{max}].$$

If $v_0 \in (v^*(m), v_{max}] \subset (v_{min}, v_{max})$, then the above identity implies that $\lambda = \lambda_*(v_0)$ and hence $\tau(\lambda, v_0) = T(v_0)$. From $v_0 > v^*(m)$ we now deduce $\tau_* = T(v_0) > \tau_0^*$ and hence we can compare (2.9) with (3.2) to deduce $x_* < 0$, a contradiction.

If $v_0 > v_{max}$, then by the monotonicity of $\tau(\cdot, \cdot)$, we deduce

$$\tau(\lambda, v_0) > \tau(\lambda, v_{max}) > \tau(0, v_{max}).$$

Therefore, recalling $\lambda_*(v_{max}) = \lambda_0(v_{\max}) = 0$, we obtain

$$\begin{aligned} m &= \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda, v_0)\max\{0, C_0 y - C_0/2\}}dy \\ &< \int_0^1 f(w_0 e^{-A\tau(0, v_{max})\max\{0, C_0 y - C_0/2\}}dy \\ &= m, \end{aligned}$$

again a contradiction. Therefore none of the cases that lead to (2.8) and (2.9) can happen. This implies that either (a2) or (b2) happens.

Next we show that case (b2) cannot happen. Otherwise, by (2.27) we obtain

$$m < f(w_0 e^{-A_0}).$$

Hence $\tau_1^* > 0$ is defined. Moreover, comparing (2.27) with (3.4) we obtain

$$\tau_* > \tau_1^*, \ \tau_*[C_0/2 + \tilde{C}(b_*)] < \tau_1^* C_0,$$

which imply, by (2.28) and (2.29) that $v_0 < v_*(m) < v^*(m)$, contradicting (3.14).

Therefore we necessarily have case (a2). We now introduce the notations

$$\lambda = \tau_* C(a_*), \ \Gamma = \tau_*[C_0/2 + C(a_*)].$$

From (2.19), (2.20) and (2.21) we find that

$$m = \int_0^1 f\big(w_0 e^{-A\max\{\lambda, \Gamma y\}}\big)dy, \tag{3.15}$$

$$v_0 \geq \frac{w_0}{\alpha} e^{-A\lambda} + \frac{\Gamma}{d_2} m(1 + \beta^{-1}), \tag{3.16}$$

with equality holding if $a_* < 0$.

Suppose now $v_0 \geq \Lambda(0)$. We claim that in this case we have $\lambda = 0$ and hence, by (3.15), $\Gamma = \Gamma(0) = \lambda^0$. Suppose for the sake of contradiction that $\lambda > 0$. From (3.15) and (3.11) we easily see that $\lambda \leq \lambda_0$. Now $C(a_*) > 0$ and hence $a_* < 0$. Thus equality in (3.16) holds. By (3.15) we deduce $\Gamma = \Gamma(\lambda)$ and hence it follows from (3.16) that $v_0 = \Lambda(\lambda) < \Lambda(0)$, contradicting our assumption on v_0 above. Hence in this case, we have $\lambda = 0$ and thus

$$C(a_*) = 0, \ \tau_* = \Gamma(0)/(C_0/2).$$

Next we suppose that $v^*(m) < v_0 < \Lambda(0)$. From (3.15) we deduce $\Gamma = \Gamma(\lambda)$ for some $\lambda \in [0, \lambda_0]$. We must have $\lambda > 0$ for otherwise, from (3.15) and (3.16) we deduce $\Gamma = \Gamma(0)$ and $v_0 \geq \Lambda(0)$, contradicting our current assumption on v_0. Therefore $\lambda > 0$ and hence $a_* < 0$, implying that equality in (3.16) holds. Recalling $\Gamma = \Gamma(\lambda)$, we thus obtain $v_0 = \Lambda(\lambda)$, and $\lambda = \lambda^*$. It follows that τ_* and a_* in Lemma 2.4 are uniquely determined by

$$\tau_* C(a_*) = \lambda^*, \ \tau_*[C_0/2 + C(a_*)] = \Gamma(\lambda^*),$$

namely

$$\tau_* = \frac{\Gamma(\lambda^*) - \lambda^*}{C_0/2}, \ a_* = C^{-1}(\lambda^*/\tau_*).$$

The rest of the proof now follows from Lemma 2.4. □

We now consider the remaining case that $\underline{v}(m) < v_0 < v_*(m)$, which can happen only if $m < f(w_0 e^{-A_0})$. Suppose that λ_0, λ^0, λ^0_* and $\Gamma(\lambda)$ are as in Theorem 3.2 but with w_0 there replaced by $w_0 e^{-A_0}$, and we denote them by $\tilde{\lambda}_0$, $\tilde{\lambda}^0$, $\tilde{\lambda}^0_*$ and $\tilde{\Gamma}(\lambda)$, respectively. Define

$$\Delta(\lambda) := \frac{w_0}{\alpha} e^{-A_0 - A\lambda} + \frac{\tilde{\Gamma}(\lambda)}{d_2} m\beta^{-1}.$$

Then $\Delta(\lambda)$ is a decreasing function over $[0, \tilde{\lambda}_0]$ with

$$\Delta(0) = \frac{w_0}{\alpha} e^{-A_0} + \frac{\tilde{\lambda}^0}{d_2} m\beta^{-1}, \ \Delta(\tilde{\lambda}^0_*) = \frac{w_0}{\alpha} e^{-A_0 - A\tilde{\lambda}^0_*} + \frac{\tilde{\lambda}^0_*}{d_2} m\beta^{-1} = v_*(m).$$

Theorem 3.3. *Suppose that* $m < f(w_0 e^{-A_0})$ *and*

$$\underline{v}(m) < v_0 < v_*(m) = \Delta(\tilde{\lambda}^0_*). \tag{3.17}$$

if $v_0 > \Delta(0)$ *and* $\lambda_* \in (0, \tilde{\lambda}^0_*)$ *is uniquely determined by* $v_0 = \Delta(\lambda_*)$, *then*

$$u_n \to 0 \ \textit{in} \ C([0, 1-\epsilon]), \ \forall \epsilon \in (0,1), \ \int_0^1 u_n dx \to \tilde{\Gamma}(\lambda_*),$$

$$v_n(x) \to v_0 - \frac{\tilde{\Gamma}(\lambda_*)}{d_2} m\beta^{-1} \ \textit{uniformly in} \ [0,1].$$

If $\underline{v}(m) < v_0 \leq \Delta(0)$, *then the above conclusions hold with* $\lambda_* = 0$.

Proof. This is similar to that of Theorem 3.2. Here we can show that case (b2) must happen, and then we use Lemma 2.5. We omit the details. □

Remark 3.4. We now compare our results with the game theoretical model in [KL], and explain the predictions that our theoretical results offer for the phytoplankton problem being modelled.

(i) Firstly we note that if we replace $\max\{C_0/2, C_0 y\}$ in (2.9) by C_0, then the system of equations for (x_*, τ_*) in Theorem 3.1 reduces to the game theoretical model of [KL], namely equations (4) and (5) in [KL] with $\hat{B} = \tau_* C_0$.

(ii) When $v_0 > v^*(m)$, from Theorem 3.2 and Step 1 of the proof of Theorems 3.1 we see that as $\sigma \to \infty$ the total biomass has limit

$$\Gamma(\lambda^*) = \tau_*[C_0/2 + C(a_*)] > \tau_0^* C_0.$$

If we have simply used (2.8) and (2.9) with $x_* = 0$ to calculate the total biomass, we would have obtained the incorrect limit $\tau_0^* C_0$. Similarly, the limit of the total biomass in the case of Theorem 3.3 is less than the value one would have obtained by simply using (2.8) and (2.9) with $x_* = 1$.

(iii) By Theorem 3.2, we find that when $v_0 \geq v^{**} := \Lambda(0)$, the limiting total biomass is a constant function of v_0, no longer increasing with v_0, while it increases with v_0 for $v_0 \in [v^*(m), v^{**}]$.

(iv) By Theorem 3.3, we find that if $m < f(w_0 e^{-A_0})$, then $\underline{v}(m) < v_{**} := \Delta(0) < m_*(m)$, and for $v_0 \in (\underline{v}(m), v_{**}]$, the limiting total biomass takes a constant value, while it is increasing with v_0 for $v_0 \in (v_{**}, v_*(m)]$.

(v) In view of points (ii), (iii) and (iv) above, our Theorems 3.1, 3.2 and 3.3 provide several new insights to the model beside confirming the predictions obtained through numerical simulation in [KL].

The important predictions in [KL] that are confirmed here are:

(1) Depth-regulating phytoplankton can form a thin layer in a poorly mixed water column, as supported by widespread empirical evidences.

(2) The concentration of the limiting nutrient should be low and constant above the phytoplankton layer and linearly increasing with depth below the layer.

The new predictions are:

(3) There are two critical levels for the nutrient concentration v_0 at the sediment: $v_* := v_*(m)$ and $v^* := v^*(m)$. The biomass layer reaches the surface when $v_0 = v^*$, and it stays at the surface for $v_0 > v^*$; the layer reaches the bottom when $v_0 = v_*$, and it stays at the bottom when $v_0 < v_{**}$.

(4) The total biomass increases with v_0 as v_0 varies between v_* and v^*; it keeps increasing when v_0 increases over the critical level v^* (so the biomass layer is

at the surface) until it reaches a second critical level v^{**}, when the biomass reaches its maximum, say B_{max}; the total biomass stays at this maximal level B_{max} for $v_0 > v^{**}$.

(5) When the death rate of the biomass is relatively small ($m < f(w_0 e^{-A_0})$), the total biomass keeps decreasing when v_0 decreases below v_* (so the biomass layer is at the bottom) until it reaches a second critical level v_{**}, where the total biomass reaches a minimal positive level, say B_{min}; the total biomass stays at this minimal level B_{min} for $v_0 < v_{**}$ until v_0 reaches its minimal possible level $v_0 = \underline{v}(m)$, then the phytoplankton biomass disappears.

(vi) We could fix v_0 and use a different parameter in the model, say the surface light level w_0, as a varying parameter to interpret the phenomena represented in items (3), (4) and (5) above.

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