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CONCENTRATION PHENOMENA IN A NONLOCAL QUASI-LINEAR PROBLEM MODELLING PHYTOPLANKTON I: EXISTENCE*

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Abstract. We study the positive steady state of a quasi-linear reaction-diffusion system in one space dimension introduced by Klausmeier and Litchman for the modelling of the distributions of phytoplankton biomass and its nutrient. The system has nonlocal dependence on the biomass function, and it has a biomass-dependent drifting term describing the active movement of the biomass towards the location of the optimal growth condition. We obtain complete descriptions of the profile of the solutions when the coefficient of the drifting term is large, rigorously proving the numerically observed phenomenon of concentration of biomass for this model. Our theoretical results reveal four critical numbers for the model not observed before and offer several further insights into the problem being modelled. This is Part I of a two-part series, where we obtain nearly optimal existence and nonexistence results. The asymptotic profile of the solutions is studied in the separate Part II.

Key words. quasi-linear, nonlocal dependence, phytoplankton, concentration phenomenon, reaction-diffusion equation

AMS subject classifications. 35J55, 35J65, 92D25

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1. Introduction.

In this paper, we study the problem

$$
\begin{cases}
-[d_1 u_x + \sigma c(x) u]_x = [g(x) - m]u, & 0 < x < 1,\\
-d_2 v_{xx} = -g(x) u, & 0 < x < 1,\\
d_1 u_x + \sigma c(x) u = 0, & x = 0, 1,\\
v_x(0) = 0, \quad v_x(1) = \beta[v_0 - v(1)], &
\end{cases}
\tag{1.1}
$$

where d_1, d_2, σ, m, v_0, and β are positive constants,

$$g(x) = f(\min\{\alpha v(x), w(x)\}), \quad f(s) = \frac{rs}{K_I + s},$$

and

$$w(x) = w_0 \exp\left[-A_0 x - A \int_0^x u(s)ds \right],$$

with α, r, K_I, w_0, A, and A_0 positive constants. We note that the right-hand sides of the differential equations in (1.1) depend on the unknown functions u and v in a nonlocal manner. Moreover, the positive function $c(x)$ is determined by u and v in a rather unconventional way to be explained below. We are interested in positive

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solutions of (1.1), namely $u > 0$ and $v > 0$ in $[0, 1]$. From (1.1) it is easy to see that for any such solution, v is an increasing function. Clearly w is a decreasing function. The function $c(x)$ is defined by

$$c(x) = \frac{x - x_0}{\delta + |x - x_0|},$$

where $\delta > 0$ is a small constant and $x_0 \in [0, 1]$ is the intersection point of the functions $\alpha u(x)$ and $w(x)$ whenever such an intersection occurs in $[0, 1]$; if $\alpha u(x)$ and $v(x)$ do not intersect, then $x_0 = 0$ if $\alpha v > w$ on $[0, 1]$, and $x_0 = 1$ if $\alpha v < w$ on [0,1]. In other words, x_0 is given by the following description:

$$\min\{\alpha v(x), w(x)\} = \alpha v(x)\ \forall x \in [0, x_0); \quad \min\{\alpha v(x), w(x)\} = w(x)\ \forall x \in (x_0, 1].$$

This unconventional dependence of c on the unknown solution (u, v) makes (1.1) a very special quasi-linear problem.

Such a system arises in the mathematical modelling of phytoplankton in a one-dimensional water column, where $u(x)$ represents the distribution of phytoplankton biomass, $v(x)$ stands for the distribution of nutrient, and x denotes the depth in the water column, with $x = 0$ at the surface and $x = 1$ at the bottom. The term $\sigma c(x)$ is used to describe the active movement of the biomass towards the spatial location with the optimal growth condition. Klausmeier and Litchman [KL] propose using this model to study the concentration phenomenon widely observed for phytoplankton in lakes and oceans. Their numerical analysis in [KL] demonstrates that for large σ, the biomass function $u(x)$ concentrates at a certain level $x = x_*$ while the nutrient function $v(x)$ is close to a piecewise linear function. They then treat u as a constant multiple of the δ-function concentrating at x_* and propose a game theoretical model to determine the location of x_*.

In this paper, we rigorously prove the existence of such a concentration phenomenon and obtain accurate formulas for the determination of x_* and the total biomass. Our theoretical results offer several further insights into the model besides those obtained through numerical analysis in [KL]; for example, we show the existence of four critical values $v_{**} < v_* < v^* < v^{**}$ for v_0 (the nutrient level at the sediment), such that

(i) $x_* = 0$ when $v_0 \geq v^*$, $x_* \in (0, 1)$ when $v_0 \in (v_*, v^*)$, and $x_* = 1$ when $v_0 \leq v_*$;

(ii) the total biomass increases with v_0 in the range $v_{**} < v_0 < v^{**}$, but it stays constant for $v_0 \geq v^{**}$ or $v_0 \leq v_{**}$ (and with v_0 above a certain level so that the biomass can survive).

It turns out that the game theoretical model of [KL] is a simplified version of our equations governing x_* and the total phytoplankton biomass for the case $v_* \leq v_0 \leq v^*$. A more detailed description of these results is given in the introduction of Part II, with their biological interpretations given in section 4 there.

To explain this model more precisely, we start by a brief description of the background and motivation of this research. Phytoplankton, the generic name of microorganisms living in lakes and oceans, is the basis of the aquatic food chain. Its importance for the proper functioning of the aquatic ecosystem has long been recognized, and its behavior has been widely studied. The distribution of phytoplankton in lakes and oceans is highly heterogeneous. To better understand this property of the phytoplankton, various mathematical models have been proposed and numerically analyzed; see, for example, [EATSH, KL, PT, PTHS, HTKS]. However, little rigorous mathematical analysis is available. In [YN], an ordinary differential equation model for the vertical distributions of phytoplankton is theoretically analyzed; see also [IT, BFH, BFHK] for earlier related research. It is our hope that the current paper may induce further rigorous mathematical research in this direction and that the techniques developed here may find more applications.

We now describe the model in more detail. In poorly mixed water columns, it has been observed that algae can be heterogeneously distributed, with thin layers of biomass on the surface, at depth, or on the sediment surface; examples for each of these cases can be found in [KL]. To model these phenomena, [KL] proposes a reaction-diffusion-taxis model of phytoplankton, nutrients, and light, based on the principle of light and nutrient competition. They use the following system to describe the distribution of phytoplankton in a one-dimensional water column, with depth represented by $0 \le z \le z_b$; $z = 0$ at the surface and $z = z_b$ at the bottom:

$$\begin{cases} b_t = D_b b_{zz} + [\nu(g^0_z)b]_z + [g^0 - m]b, & 0 < z < z_b, t > 0, \\ R_t = D_R R_{zz} - bg^0/Y + \epsilon mb/Y, & 0 < z < z_b, t > 0, \\ D_b b_z + \nu(g^0_z)b = 0, & z = 0, z_b, t > 0, \\ R_z(t,0) = 0, R_z(t,z_b) = h[R_{in} - R(t,z_b)], & t > 0, \\ I(t,z) = I_{in} \exp\left(- \int_0^z [ab(t,s) + a_{bg}]ds \right), & 0 < z < z_b, t > 0, \end{cases} \tag{1.2}$$

where a, a_{bg}, h, D_b, D_R, m, I_{in}, R_{in}, and Y are positive constants, $\epsilon \in [0,1)$, $\nu(s)$ is an odd decreasing function that approaches $\nu_{max} > 0$ as $s \to -\infty$, and

$$g^0(t,z) = \min\{f_I(I(t,z)), f_R(R(t,z))\},$$

with

$$f_I(s) = r\frac{s}{s+K_I}, \quad f_R(s) = r\frac{s}{s+K_R}, \quad r, K_I, K_R > 0.$$

In (1.2), $b(t,z)$ denotes the distribution of the phytoplankton biomass, $R(t,z)$ represents the nutrient distribution, and $I(t,z)$ stands for the distribution of light. The constant I_{in} is the light distribution at the surface, and, by the Lambert–Beer law, light at depth z is given by

$$I(t,z) = I_{in} \exp\left(- \int_0^z [ab(t,s) + a_{bg}]ds \right),$$

where a and a_{bg} are, respectively, the phytoplankton and background attenuation coefficients. In this model, it is assumed that the change in phytoplankton biomass at depth z results from three processes: growth, loss, and movement. The functions $f_I(I)$ and $f_R(R)$ are the phytoplankton growth rate when only one of the resources I and R is limited (the other being regarded as sufficient). By Liebig's law of the minimum for essential resources, the gross phytoplankton growth rate is given by $g^0(t,z) = \min\{f_I(I(t,z)), f_R(R(t,z))\}$. Biomass is lost at density-dependent rate m, representing respiration, death, and grazing. D_b is the passive diffusion rate of the biomass, while $[\nu(g^0_z)b]_z$ describes active movement of the biomass towards a spatial location (i.e., depth) with a better growth condition. The no-flux boundary condition for b means that no phytoplankton enters or leaves the water column at $z = 0$ and $z = z_b$. The equation for R is based on the assumption that nutrients in the water column are mixed with eddy diffusion with diffusion coefficient D_R and are consumed by phytoplankton at the rate $-bg^0/Y$, and the term $\epsilon mb/Y$ means that ϵ proportion of the nutrients in dead phytoplankton is immediately recycled. Here Y describes the yield of phytoplankton biomass per unit nutrient consumed. The boundary condition for R means that nutrients do not leave or enter the top of the water column but are supplied at the bottom, with nutrients in the sediments fixed at constant concentration R_{in}, which diffuse across the sediment-water interface at a rate proportional to the concentration difference across the interface; the parameter h describes the permeability of the interface.

In [KL], taking $\epsilon = 0$ and $\nu(s) = \nu_0(s) := -\nu_{max}\text{sgn}(s)$ (where $\text{sgn}(s)$ is the sign function, which equals 1, -1, or 0 according to whether $s > 0$, $s < 0$, or $s = 0$), the equilibrium distributions of b, R, and I are calculated numerically for various parameter values (see Table 1 and Figure 1 in [KL]). The numerical simulation in [KL] shows that as ν_{max} increases, the biomass distribution concentrates at a certain depth $z = z^*$. Further, based on intuition and formal analysis, a game theoretical approach is proposed in [KL], which can be used to calculate z^*. Though the connection between (1.2) and the simplified game theoretical approach is not rigorously established, the predictions deduced from the game theoretical model agree well with the numerical results based on (1.2); see details in [KL].

In this paper, we theoretically analyze the equilibrium solutions of (1.2). So $b = b(z)$, $R = R(z)$, and $I = I(z)$. Naturally, only positive solutions are of interest to us.

As in [KL], we assume that $\epsilon = 0$. We denote $f(s) = rs/(s+K_I)$ and $\alpha = K_I/K_R$. Then

$$f_I(s) = f(s), \quad f_R(s) = f(\alpha s).$$

Since $f'(s) > 0$ we find that

$$g^0(z) = \min\{f(I(z)), \quad f(\alpha R(z))\} = f(\min\{I(z), \alpha R(z)\}).$$

Clearly $I(z)$ is a decreasing function. Since $D_R R'' = bg^0/Y > 0$ and $R'(0) = 0$, we find that $R'(z) > 0$ for $z \in (0, z_b]$. Therefore $R(z)$ is increasing, and we can always find a unique $z_0 \in [0, z_b]$ such that

$$g^0(z) = f(\alpha R(z)) \; \forall z \in [0, z_0), \quad g^0(z) = f(I(z)) \; \forall z \in (z_0, z_b].$$

Evidently z_0 depends on I and R, and $z_0 = 0$ if $\alpha R(z) \geq I(z)$ on $[0, z_b]$, and $z_0 = z_b$ when $\alpha R(z) \leq I(z)$ on $[0, z_b]$.

In view of the above discussions for $g^0(z)$, we see that

$$\nu_0(g_z^0(z)) = -\nu_{max}\text{sgn}(g_z^0(z)) = -\nu_{max} \; \forall z \in [0, z_0),$$
$$\nu_0(g_z^0(z)) = -\nu_{max}\text{sgn}(g_z^0(z)) = \nu_{max} \; \forall z \in (z_0, z_b].$$

In this paper, we use a continuous approximation of the above step function used in [KL]; namely, we take

$$\nu(g_z^0(z)) = \nu_{\delta'}(z) := \nu_{max}\frac{z - z_0}{\delta' + |z - z_0|}.$$

It is easily seen that $\nu_{\delta'}(z) \to \nu_0(g_z^0(z))$ as $\delta' \to 0$. We stress again that $\nu_{\delta'}$ depends on I and R through the definition of z_0.

Next we normalize the functions in (1.2) by

$$u(x) = b(z_b x)/Y, \quad v(x) = R(z_b x), \quad w(x) = I(z_b x), \quad 0 \le x \le 1,$$

and define

$$d_1 = z_b^2 D_b, \quad d_2 = z_b^2 D_R, \quad \sigma = \nu_{max} z_b, \quad \delta = \delta'/z_b,$$

$$A = a z_b Y, \quad A_0 = a_{bg} z_b, \quad \beta = h z_b, v_0 = R_{in}, \quad w_0 = I_{in}.$$

We denote

$$c(x) = c_{v,w}(x) = \frac{x - x_0}{\delta + |x - x_0|}, \quad x_0 = \frac{z_0}{z_b}.$$

Then after some simple calculations we find that the steady-state version of (1.2) becomes (1.1), or, written in a more comprehensive form,

$$\begin{cases} -[d_1 u_x + \sigma c(x) u]_x = [f(\min\{\alpha v, w\}) - m]u, & 0 < x < 1, \\ -d_2 v_{xx} = -f(\min\{\alpha v, w\})u, & 0 < x < 1, \\ d_1 u_x(0) + \sigma c(0) u(0) = d_1 u_x(1) + \sigma c(1) u(1) = 0, & \\ v_x(0) = 0, \quad v_x(1) = \beta[v_0 - v(1)], & \\ w(x) = w_0 \exp\left[-A_0 x - A \int_0^x u(s) ds \right], & 0 \le x \le 1. \end{cases} \tag{1.3}$$

Let us note that from the equation for u and the strong maximum principle, if u is nonnegative on $[0, 1]$, then it is either identically 0 or positive everywhere in $[0, 1]$. It is also easy to see that whenever u is positive, $0 < v < v_0$ in $[0, 1]$.

Since this paper is very long, and the techniques used in the first half of the paper are rather different from those in the second half, we divide it into two separate parts. Part I here is mainly concerned with the existence and nonexistence problem, and Part II studies the asymptotic behavior of the positive solutions as $\sigma \to \infty$.

In section 2, treating m as a parameter, we make use of a bifurcation argument to obtain two critical numbers $0 < m_* \le m^*$ such that (1.3) has no positive solution when $m \ge m^*$ and it has at least one positive solution when $0 < m < m_*$. We also show that as m (the death rate of the biomass) decreases to 0, the biomass blows up everywhere; the exact limiting profiles of the biomass function and the nutrient function as $m \to 0$ are also obtained. In section 3, we show that as $\sigma \to \infty$, m_* and m^* converge to the same limit $f(\min\{\alpha v_0, w_0\})$. This demonstrates that our existence and nonexistence results are sharp for large σ.

The asymptotic behavior of the positive solutions when $\sigma \to \infty$ is investigated separately in Part II (see [DH]), where we fix $0 < m < f(\min\{\alpha v_0, w_0\})$ and study the asymptotic behavior of a positive solution (u_n, v_n) of (1.3) with $\sigma = \sigma_n \to \infty$.

2. Existence and nonexistence results. The function $c(x)$ appearing in (1.3) plays a very important role in our analysis. From the definition of $c(x) = c_{v,w}(x)$ we find that it is well defined if $v(x)$ is increasing in $[0, 1]$ and $w(x)$ is decreasing in $[0, 1]$, and

$$c(x) = \frac{x - x_0}{\delta + |x - x_0|},$$

where $x_0 \in [0,1]$ is uniquely determined by the following:

$$\min\{\alpha v(x), w(x)\} = \alpha v(x)\ \forall x \in [0, x_0); \quad \min\{\alpha v(x), w(x)\} = w(x)\ \forall x \in (x_0, 1].$$

It is easily seen that for the definition of $c_{v,w}(x)$, the requirement that v is increasing and w is decreasing can be relaxed; we can allow one (but not both) of the following:

(i) v is nondecreasing, (ii) w is nonincreasing.

Let us also observe that $c(x)$ is a C^1 function, with $c'(x) = \delta(\delta + |x - x_0|)^{-2}$.

With this in mind, we find that $(u, v, w) = (0, v_0, w_*)$ solves (1.3), where $w_*(x) = w_0 e^{-A_0 x}$. We will call this the trivial solution. To find nontrivial solutions, we now treat m as a parameter and look for special values of m so that positive solutions of (1.3) may bifurcate from this trivial solution. If $m_* \geq 0$ is such a value, then there exist $m_n \to m_*$ and (u_n, v_n, w_n) solving (1.3) with $m = m_n$ such that $u_n > 0$, $v_n > 0$, and $u_n \to 0$, $v_n \to v_0$, and $w_n \to w_*$ and in $C^1[0,1]$ as $n \to \infty$. Now $v_n(x)$ is increasing and $w_n(x)$ is decreasing for $x \in [0,1]$; therefore $c_{v_n,w_n}(x)$ is well defined. Moreover, it is easily checked that $c_{v_n,w_n} \to c_{v_0,w_*}$ in $C([0,1])$ as $n \to \infty$. To simplify the notation, we write $c^n(x) = c_{v_n,w_n}(x)$ and $c^0(x) = c_{v_0,w_*}(x)$. Therefore u_n satisfies

$$\begin{cases} -[d_1 u_n' + \sigma c^n(x) u_n]' = [f(\min\{\alpha v_n, w_n\}) - m_n] u_n & \text{in } (0,1), \\ d_1 u_n' + \sigma c^n(x) u_n = 0 & \text{for } x = 0, 1. \end{cases}$$

Here and in what follows, we use the notation $u' = u_x$, etc. To determine the value of m_*, we first deduce a useful equation from the equation for u_n. So we define $\hat{u}_n = u_n / \|u_n\|_\infty$. Then we have

$$\begin{cases} -(d_1 \hat{u}_n' + \sigma c^n \hat{u}_n)' + m_n \hat{u}_n = f(\min\{\alpha v_n, w_n\}) \hat{u}_n & \text{in } (0,1), \\ d_1 \hat{u}_n' + \sigma c^n \hat{u}_n = 0 & \text{for } x = 0, 1. \end{cases} \tag{2.1}$$

Since the right-hand side of the first equation in (2.1) and $\{\hat{u}_n\}$ are both bounded in $L^\infty([0,1])$, and since m_n, c^n, $(c^n)'$ are bounded in $L^\infty([0,1])$, we can use standard L^p theory for elliptic operators (see [GT]) to conclude that $\{\hat{u}_n\}$ is a bounded sequence in $W^{2,p}([0,1])$ for any $p > 1$. By the Sobolev embedding theorem, we see that $\{\hat{u}_n\}$ is compact in $C^1([0,1])$. By passing to a subsequence, we may assume that $\hat{u}_n \to \hat{u}$ in $C^1([0,1])$, and then we easily see that $\hat{u}$ satisfies (in the weak sense)

$$\begin{cases} -(d_1 \hat{u}' + \sigma c^0 \hat{u})' + m_* \hat{u} = f(\min\{\alpha v_0, w_*\}) \hat{u} & \text{in } (0,1), \\ d_1 \hat{u}' + \sigma c^0 \hat{u} = 0 & \text{for } x = 0, 1. \end{cases} \tag{2.2}$$

Since $\hat{u} \geq 0$ and $\|\hat{u}\|_\infty = 1$, we necessarily have, by applying the strong maximum principle to (2.2), that $\hat{u} > 0$. This implies that $-m_*$ is the principal eigenvalue of the problem

$$\begin{cases} -(d_1 u' + \sigma c^0 u)' - f(\min\{\alpha v_0, w_*\}) u = \lambda u & \text{in } (0,1), \\ d_1 u' + \sigma c^0 u = 0 & \text{for } x = 0, 1. \end{cases} \tag{2.3}$$

One easily checks that 0 is the first eigenvalue of the problem

$$\begin{cases} -(d_1 u' + \sigma c^0 u)' = \lambda u & \text{in } (0,1), \\ d_1 u' + \sigma c^0 u = 0 & \text{for } x = 0, 1. \end{cases}$$

Since $-f(\min\{\alpha v_0, w_*\}) < 0$, by the characterization of the first eigenvalues (see, for example, Theorems 2.4 and 2.8 of [D]), the first eigenvalue of (2.3) is less than 0, and hence $m_* > 0$. On the other hand, if (u, v, w) is a positive solution to (1.3), then rewriting the equation for u in the form

$$\begin{cases} -(d_1 u' + \sigma c u)' - f(\min\{\alpha v, w\})u = -mu & \text{in } (0,1), \\ d_1 u' + \sigma c u = 0 & \text{for } x = 0, 1, \end{cases}$$

we find that $-m$ is the first eigenvalue of the problem

$$\begin{cases} -(d_1 u' + \sigma c u)' - f(\min\{\alpha v, w\})u = \lambda u & \text{in } (0,1), \\ d_1 u' + \sigma c u = 0 & \text{for } x = 0, 1, \end{cases}$$

which we denote by $\lambda_1(c, v, w)$. Clearly $v < v_0$ and $w < w_*$ in $(0,1)$. It follows that $-f(\min\{\alpha v, w\}) > -f(\min\{\alpha v_0, w_*\})$ in $(0,1)$ and hence

$$\lambda_1(c, v, w) > \lambda_1(c, v_0, w_*). \tag{2.4}$$

In this notation, clearly $-m_* = \lambda_1(c^0, v_0, w_*)$. Since $c(x) = (x - x_0)/(\delta + |x - x_0|)$ is determined completely by x_0, it is convenient to introduce the notation

$$C_{x_0} = \frac{x - x_0}{\delta + |x - x_0|}$$

and

$$m^* = -\inf_{x_0 \in [0,1]} \lambda_1(C_{x_0}, v_0, w_*).$$

It is easy to show that m^* is achieved by some $x_0 \in [0,1]$, and $m^* \geq m_*$. Moreover, from the above discussion, we have the following result.

PROPOSITION 2.1. *If* (1.3) *has a positive solution, then necessarily* $m < m^*$.

We will show that (1.3) has a positive solution for every $0 < m < m_*$. Before that we briefly discuss some further simple estimates for the values of m so that (1.3) has a positive solution. So suppose that (1.3) has a positive solution (u, v, w). Integrating the equation for u over $[0,1]$, we obtain

$$\int_0^1 \Big[f(\min\{\alpha v, w\}) - m\Big] u dx = 0.$$

Since $u > 0$, $[f(\min\{\alpha v, w\}) - m]$ must change sign over $(0,1)$, and therefore

$$\min_{[0,1]} f(\min\{\alpha v(x), w(x)\}) < m < \max_{[0,1]} f(\min\{\alpha v(x), w(x)\}).$$

It follows that

$$f(\min\{\alpha v(0), w(1)\}) < m < f(\min\{\alpha v_0, w_0\}). \tag{2.5}$$

From a similar consideration, we have

$$m^*, m_* \in \big(f(\min\{\alpha v_0, w_*(1)\}), f(\min\{\alpha v_0, w_0\})\big). \tag{2.6}$$

We now use a global bifurcation argument to show that (1.3) has a positive solution for every $m \in (0, m_*)$. First, we transform (1.3) into an abstract nonlinear equation. Due to its unconventional nature, we cannot use a simple inverse operator trick to do this. In fact, to cope with the rather implicit dependence of $c_{v,w}$ on (v, w), in the following, we have to choose the function spaces for the abstract setting very carefully and then analyze the properties of the abstract operator mostly by definitions.

Fix $\gamma \in (0, 1)$ and set

$$K := \{\phi \in C^{1,\gamma}([0, 1]) : \phi \text{ is nondecreasing in } [0, 1]\},$$
$$P := \{\phi \in C^{1,\gamma}([0, 1]) : \phi \text{ is nonnegative in } [0, 1]\}.$$

Clearly they are closed convex sets in $C^{1,\gamma}([0, 1])$, and, moreover, P is a positive cone.

For given $(u, v) \in P \times K$ and $m \geq 0$, we define

$$w = w_0 \exp\left[- A_0 x - A \int_0^x u(s)ds\right],$$

$$c(x) = c_{v,w}(x), \quad v_+ = \max\{v, 0\}$$

and will use the solutions of the following problems to define an abstract operator $T(m, u, v)$ such that $T(m, u, v) = (u, v)$ for $(u, v) \in P \times K$ if and only if (u, v) is a nonnegative solution of (1.3). So we consider the problems

$$\begin{cases} -(d_1\phi' + \sigma c\phi)' + (m + 1)\phi = f(\min\{\alpha v_+, w\})u + u, & 0 < x < 1, \\ d_1\phi' + \sigma c\phi = 0, & x = 0, 1, \end{cases} \tag{2.7}$$

and

$$\begin{cases} -d_2\psi'' = -f(\min\{\alpha v_+, w\})u, & 0 < x < 1, \\ \psi'(0) = 0, \quad \psi'(1) = \beta[v_0 - \psi(1)]. \end{cases} \tag{2.8}$$

Clearly (2.7) has a unique solution ϕ, and it is nonnegative. Let $\zeta(x) = v_0 - \psi(x)$. Then (2.8) becomes

$$\begin{cases} -d_2\zeta'' = f(\min\{\alpha v_+, w\})u, & 0 < x < 1, \\ \zeta'(0) = 0, \quad \zeta'(1) + \beta\zeta(1) = 0. \end{cases} \tag{2.9}$$

It is easily seen that (2.9) has a unique solution ζ, and it is nonnegative. Moreover, from $\zeta'' \leq 0$, and $\zeta'(0) = 0$ we deduce that ζ is nonincreasing. Hence ψ is nondecreasing and $\psi(x) \leq v_0$ in $[0, 1]$. Thus the solution operator $(\phi, \psi) = T(m, u, v)$ is well defined for $(u, v) \in P \times K$ and $m \geq 0$, and $T(m, \cdot, \cdot)$ maps $P \times K$ into itself.

We show next that T is continuous. Suppose that $(m_n, u_n, v_n) \in [0, \infty) \times P \times K$, $m_n \to m$, and $(u_n, v_n) \to (u, v)$ in $C^{1,\gamma}([0, 1]) \times C^{1,\gamma}([0, 1])$. Then it is easily checked that $c^n := c_{v_n,w_n} \to c := c_{v,w}$ in $C^1([0, 1])$. Denote $(\phi_n, \psi_n) = T(m_n, u_n, v_n)$ and $\zeta_n = v_0 - \psi_n$. We have

$$\begin{cases} -(d_1\phi_n' + \sigma c^n\phi_n)' + (m_n + 1)\phi_n = f(\min\{\alpha(v_n)_+, w_n\})u_n + u_n, & 0 < x < 1, \\ d_1\phi_n' + \sigma c^n\phi_n = 0, & x = 0, 1, \end{cases} \tag{2.10}$$

and

$$
(2.11)\qquad \begin{cases} -d_2\zeta_n'' = f(\min\{\alpha(v_n)_+, w_n\})u_n, & 0<x<1,\\ \zeta_n'(0)=0, \quad \zeta_n'(1)+\beta\zeta_n(1)=0. \end{cases}
$$

Applying standard L^p theory to both (2.10) and (2.11), we find that $\{\phi_n\}$ and $\{\zeta_n\}$ are bounded in $W^{2,p}([0,1])$ for all $p>1$. Hence they are precompact in $C^{1,\gamma}([0,1])$. This implies that by passing to a subsequence, $\phi_n \to \phi$ and $\psi_n \to \psi$ in $C^{1,\gamma}([0,1])$. Moreover, letting $n\to\infty$ in (2.10) and (2.11) we find that necessarily $(\phi,\psi)=T(m,u,v)$. Therefore the entire original sequence converges with limit (ϕ,ψ). This proves the continuity of T.

We further show that T is compact. Suppose that $\{(m_n,u_n,v_n)\}\subset[0,\infty)\times P\times K$ is bounded. Then along some subsequence n_k, (u_n,v_n) converges in the $C^1([0,1])\times C^1([0,1])$ norm to some (u,v), and, by passing to a further subsequence, we may assume that $m_{n_k}\to m$. We may now repeat the arguments in the above continuity proof to conclude that $T(m_{n_k},u_{n_k},v_{n_k})\to T(m,u,v)$ in $C^{1,\gamma}([0,1])$. Therefore T is a compact operator on $[0,\infty)\times P\times K$.

Suppose that $m\ge 0$ and $(u,v)\in P\times K$ satisfies $(u,v)=T(m,u,v)$. We claim that $v\ge 0$ in $[0,1]$. Otherwise, due to the monotonicity of v there exists $x_0\in(0,1]$ such that $v<0$ in $[0,x_0)$ and $v\ge 0$ in $(x_0,1]$. Therefore, by (2.8),

$$
-v''=0 \text{ in } (0,x_0).
$$

Since $v'(0)=0$, we deduce that $v'(x)=0$ in $(0,x_0)$, and hence v is a negative constant in $(0,x_0)$, say $v=-c$. This is possible only if $x_0=1$ (otherwise, v is discontinuous at $x=x_0$ since $v\ge 0$ in $(x_0,1]$), but then from $v'(1)=\beta[v_0-v(1)]$ we deduce that $-c=v_0>0$, a contradiction. Therefore we must have $v\ge 0$ in $[0,1]$, as claimed. Thus (u,v) is a nonnegative solution of (1.3) if and only if $(u,v)\in P\times K$ and $T(m,u,v)=(u,v)$.

In order to apply the global bifurcation theory to the operator equation

$$
(u,v)-T(m,u,v)=0,
$$

we now calculate the Fréchet derivative of T with respect to (u,v) at $(m,0,v_0)$, in the convex set $P\times K$ of $C^{1,\gamma}([0,1])\times C^{1,\gamma}([0,1])$, where $m\ge 0$. From (2.7) and (2.8) we easily see that $T(m,0,v_0)=(0,v_0)$. For $(u,v)\in C([0,1])\times C([0,1])$, we define $(\xi,\eta)=L_m(u,v)$ to be the unique solution of the following linear problems:

$$
(2.12)\qquad \begin{cases} -(d_1\xi'+\sigma c^0\xi)'+(m+1)\xi = f(\min\{\alpha v_0, w_*\})u+u, & 0<x<1,\\ d_1\xi'+\sigma c^0\xi=0, & x=0,1, \end{cases}
$$

$$
(2.13)\qquad \begin{cases} -d_2\eta'' = -f(\min\{\alpha v_0, w_*\})u, & 0<x<1,\\ \eta'(0)=0, \quad \eta'(1)+\beta\eta(1)=0, \end{cases}
$$

where c^0 and w_* are defined as at the beginning of this section.

Suppose $(u_n,v_n)\to(0,v_0)$ in $P\times K$. Denote

$$
(\phi_n,\psi_n)=T(m,u_n,v_n) \text{ and } (\tau_n,\theta_n)=L_m(u_n,v_n-v_0).
$$

We want to show that

$$
(2.14)\qquad \|(\phi_n,\psi_n)-(0,v_0)-(\tau_n,\theta_n)\| = o(\|(u_n,v_n-v_0)\|),
$$

where $\|(u,v)\| = \max\{\|u\|, \|v\|\}$, and $\|u\| = \|u\|_{C^{1,\gamma}([0,1])}$. This would imply that the Fréchet derivative of T with respect to (u,v) at $(m,0,v_0)$, in the convex set $P \times K$, is the linear operator L_m.

Suppose $(u_n, v_n) \to (0, v_0)$ in $P \times K$. Without loss of generality we assume that $u_n \not\equiv 0$. We define w_n and $c^n = c_{v_n,w_n}$ as before, and let $\hat{\phi}_n = \phi_n/\|u_n\|$, $\hat{u}_n = u_n/\|u_n\|$. Then

$$\tag{2.15} \begin{cases} -(d_1\hat{\phi}_n' + \sigma c^n \hat{\phi}_n)' + (m+1)\hat{\phi}_n = f(\min\{\alpha(v_n)_+, w_n\})\hat{u}_n + \hat{u}_n, & 0 < x < 1, \\ d_1\hat{\phi}_n' + \sigma c^n \hat{\phi}_n = 0, & x = 0, 1. \end{cases}$$

Since the right-hand side of the first equation in (2.15) is bounded in $L^\infty([0,1])$, much as before we deduce from the L^p theory and the Sobolev imbedding theorem that there exists some positive constant C independent of n such that

$$\|\hat{\phi}_n\| \le C \;\forall n \ge 1.$$

We now define $\Phi_n = (\phi_n - \tau_n)/\|u_n\|$, and from the equations for ϕ_n and τ_n we obtain

$$\tag{2.16} \begin{cases} -(d_1\Phi_n' + \sigma c^0 \Phi_n)' + (m+1)\Phi_n = f_n, & 0 < x < 1, \\ d_1\Phi_n' + \sigma c^0 \hat{\Phi}_n = g_n, & x = 0, 1, \end{cases}$$

where

$$f_n := [f(\min\{\alpha(v_n)_+, w_n\}) - f(\min\{\alpha v_0, w_*\})]\hat{u}_n + [\sigma(c^n - c^0)\hat{\phi}_n]'$$

and

$$g_n = \sigma(c^0 - c^n)\hat{\phi}_n.$$

It is easy to check that f_n converges to 0 in $L^\infty([0,1])$, and g_n converges to 0 in $C^1([0,1])$. Since $(m+1) \ge 1$, we may apply the L^p estimate to (2.16) to conclude that $\|\Phi_n\| \to 0$ as $n \to \infty$. Hence

$$\|\phi_n - \tau_n\| = o(\|u_n\|).$$

Define $\Psi_n = [(v_0 - \psi_n) + \theta_n]/\|u_n\|$. Then from the equations for ψ_n and θ_n we deduce that

$$\tag{2.17} \begin{cases} -d_2\Psi_n'' = [f(\min\{\alpha(v_n)_+, w_n\}) - f(\min\{\alpha v_0, w_*\})]\hat{u}_n, & 0 < x < 1, \\ \Psi_n'(0) = 0, \quad \Psi_n'(1) + \beta\Psi_n(1) = 0. \end{cases}$$

Since the right-hand side of the first equation in (2.17) converges to 0 in $L^\infty([0,1])$, we can apply the L^p theory to (2.17) to conclude that $\|\Psi_n\| \to 0$ as $n \to \infty$. Therefore

$$\|\psi_n - v_0 - \theta_n\| = o(\|u_n\|).$$

Thus we have

$$\|(\phi_n, \psi_n) - (0, v_0) - (\tau_n, \theta_n)\| = o(\|(u_n, v_n - v_0)\|).$$

Summarizing the above discussions, we have the following result.

PROPOSITION 2.2. *The operator* $T : [0,\infty) \times P \times K \to P \times K$ *is completely continuous, and it is Fréchet differentiable at* $(m, 0, v_0)$ *with respect to* (u, v) *in the convex set* $P \times K$*, with derivative operator* L_m*. Moreover,* $(u, v) = T(m, u, v)$ *implies that* $v \in P$*;* (u, v) *is a nonnegative solution of* (1.3) *if and only if* $(u, v) = T(m, u, v)$*.*

We are now ready to prove the main result of this section.

THEOREM 2.3. *For every* $m \in (0, m_*)$*, problem* (1.3) *has at least one positive solution. Moreover, if* m_n *decreases to* 0 *and* (u_n, v_n) *is a positive solution of* (1.3) *with* $m = m_n$*, then* $u_n \to \infty$ *uniformly in* $[0, 1]$ *and there exists a unique* $\tau \in \big(0, \frac{v_0}{1+\beta^{-1}}\big)$ *(determined by* (2.27) *below) such that*

$$\frac{u_n(x)}{\|u_n\|_\infty} \to \Big(1 + \frac{x}{\delta}\Big)^{\frac{\sigma}{d_1}\delta} e^{-\frac{\sigma}{d_1}x}, \quad v_n(x) \to \tau x + v_0 - \tau(1 + \beta^{-1})$$

uniformly in $[0, 1]$*. Furthermore, for each* $m \in (0, m_*)$*, there is a positive solution* (m, u, v) *lying on the global bifurcation branch,* $\Gamma = \{(m, u, v)\} \subset (0,\infty) \times C^{1,\gamma}([0, 1]) \times C^{1,\gamma}([0, 1])$*, bifurcating from the trivial solution branch* $\Gamma_0 := \{(m, 0, v_0) : m \in (-\infty,\infty)\}$ *at* $m = m_*$*.*

Proof. Since the proof is rather long, we divide it into several steps.

Step 1: *Existence of an unbounded global solution branch.*

We observe that 1 is an eigenvalue of L_{m_*} with eigenvector (ϕ_1, ψ_1) satisfying $\phi_1 > 0$ and $\psi_1 < 0$. Indeed, $\phi_1 > 0$ is a principal eigenfunction of (2.3) with $\lambda = -m_*$, and ψ_1 is the unique solution of (2.13) with $u = \phi_1$, and hence $\psi_1 < 0$. In order to apply the abstract global bifurcation theory in positive cones, we define $S : [0,\infty) \times P \times (-K) \to P \times (-K)$ by

$$S(m, u, \xi) = (\phi, \zeta) \text{ if and only if } T(m, u, v_0 - \xi) = (\phi, v_0 - \zeta).$$

Then from the properties of T we find that S is completely continuous. Moreover, if we denote by $DS(m, 0, 0)$ the Fréchet derivative of S with respect to (u, ξ) in $P \times (-K)$ at $(u, \xi) = (0, 0)$, then 1 is an eigenvalue of $DS(m_*, 0, 0)$ with eigenvector $(\phi_1, -\psi_1)$, where ϕ_1 and ψ_1 are as given above. Let us denote by P_0 the nonnegative functions in $(-K)$. Clearly P_0 is a cone in $C^{1,\gamma}([0, 1])$, and hence $P \times P_0$ is a cone in $C^{1,\gamma}([0, 1]) \times C^{1,\gamma}([0, 1])$. Moreover, it is easy to check through the definition of T that $S(m, \cdot, \cdot)$ maps $P \times P_0$ into itself, and 1 is the only eigenvalue of $DS(m_*, 0, 0)$ with an eigenvector in $P \times P_0$, and for any $m \geq 0$, $m \neq m_*$, 1 is not an eigenvalue of $DS(m, 0, 0)$ corresponding to an eigenvector in $P \times P_0$. These properties allow us to apply Corollary 18.4 of [A] to conclude that there exists a global unbounded branch of solutions of $(u, \zeta) = S(m, u, \zeta)$ in $R^1 \times \big(P \times P_0 \setminus \{(0, 0)\}\big)$. We denote this global branch by $\tilde{\Gamma} = \{(m, u, \zeta)\}$ and define

$$\Gamma := \{(m, u, v_0 - \zeta) : (m, u, \zeta) \in \tilde{\Gamma}\}.$$

Clearly Γ is a global branch of solutions to $(u, v) = T(m, u, v)$ with $u \geq 0$ and $v = v_0 - \zeta \leq v_0$. We claim that $u \neq 0$. Otherwise, $u = 0$ and we deduce that $v = v_0 - \zeta = v_0$. Hence $\zeta = 0$, contradicting the fact that $(m, u, \zeta) \in R^1 \times \big(P \times P_0 \setminus \{(0, 0)\}\big)$. Hence $u \geq 0$ and $u \neq 0$. It then follows from Proposition 2.2 that $v_0 - \zeta \geq 0$. But $u \neq 0$ implies that $v_0 - \zeta \neq 0$, and hence $(u, v_0 - \zeta)$ is a positive solution of (1.3). Thus we have proved that Γ is an unbounded branch of positive solutions of (1.3).

Step 2: *We show that the* m*-range of* Γ *covers* $(0, m_*)$*.*

If $(m, u, v) \in \Gamma$, then from Proposition 2.1 and (2.5) we deduce that $0 < m < m^*$. Therefore we can find a sequence $(m_n, u_n, v_n) \in \Gamma$ such that $m_n \to m_0 \in [0, m^*]$ and $\|(u_n, v_n)\| \to \infty$. Note that $c_{v_n, w_n} = C_{x_n}$ for some $x_n \in [0,1]$ uniquely determined by v_n and w_n. By passing to a subsequence we may assume that $x_n \to x_0 \in [0,1]$. Then it is easily seen that $C_{x_n} \to C_{x_0}$ in $C^1([0,1])$. We necessarily have, by passing to a subsequence, that $\|u_n\|_\infty \to \infty$, for otherwise from the equation for u_n we can deduce that $\|u_n\|$ is bounded, which in turn implies that $\|v_n\|$ is bounded, contradicting our assumption that $\|(u_n, v_n)\| \to \infty$. Therefore we may assume that $\|u_n\|_\infty \to \infty$. Denote $\hat{u}_n = u_n/\|u_n\|_\infty$. Then we can use the L^p estimate to the equation for $\hat{u}_n$ to deduce that $\{\hat{u}_n\}$ is precompact in $C^{1,\gamma}([0,1])$. Hence we may assume that $\hat{u}_n \to \hat{u}$ in $C^{1,\gamma}([0,1])$. Since $0 \le f(\min\{\alpha v_n, w_n\}) \le f(w_*)$, we may assume that $f(\min\{\alpha v_n, w_n\})$ converges to f_0 weakly in $L^2([0,1])$. Clearly we also have $0 \le f_0 \le f(w_*)$. Passing to the weak limit in the equation for $\hat{u}_n$ we deduce that $\hat{u}$ is a weak solution of

$$\text{(2.18)} \qquad \begin{cases} -(d_1 \hat{u}' + \sigma C_{x_0} \hat{u})' = (f_0 - m_0)\hat{u}, & 0 < x < 1, \\ \quad d_1 \hat{u}' + \sigma C_{x_0} \hat{u} = 0, & x = 0, 1. \end{cases}$$

Since $\hat{u} \ge 0$ and $\|\hat{u}\|_\infty = 1$, we can apply the Harnack inequality and the strong maximum principle to (2.18) to conclude that $\hat{u} > 0$ in $[0,1]$. This implies that $u_n = \|u_n\|_\infty \hat{u}_n \to \infty$ uniformly in $[0,1]$. Therefore $w_n \to 0$ uniformly on any compact subset of $(0,1]$. This implies that $f_0 = 0$, and hence we deduce from (2.18) that $-m_0$ is the first eigenvalue of

$$\text{(2.19)} \qquad \begin{cases} -(d_1 u' + \sigma C_{x_0} u)' = \lambda u, & 0 < x < 1, \\ \quad d_1 u' + \sigma C_{x_0} u = 0, & x = 0, 1. \end{cases}$$

Hence $m_0 = 0$ and

$$\hat{u} = \exp\left[-\frac{\sigma}{d_1} \int_0^x C_{x_0}(s) ds \right].$$

This implies that the entire original sequence $\{m_n\}$ converges to 0. By the connectedness of Γ, we can conclude that for every $m \in (0, m_*)$, (1.3) has at least one positive solution lying on Γ. Moreover, when $x_n \to x_0$, we have

$$\frac{u_n}{\|u_n\|_\infty} \to \exp\left[-\frac{\sigma}{d_1} \int_0^x C_{x_0}(s) ds \right].$$

Step 3: *The limiting profile of* u_n.

We will show in a moment that $x_0 = 0$ and hence the entire original sequence $u_n/\|u_n\|_\infty$ converges in $C^{1,\gamma}([0,1])$ to $\exp\left[-\frac{\sigma}{d_1} \int_0^x C_0(s) ds \right]$.

Let $\zeta_n = v_0 - v_n$. Then

$$\text{(2.20)} \qquad \begin{cases} -d_2 \zeta_n'' = f(\min\{\alpha v_n, w_n\}) u_n, & 0 < x < 1, \\ \zeta_n'(0) = 0, \quad \zeta_n'(1) + \beta \zeta_n(1) = 0. \end{cases}$$

We have

$$0 \le f(\min\{\alpha v_n, w_n\}) u_n \le f(w_n(x)) u_n(x).$$

Moreover,

$$\begin{aligned} f(w_n(x))u_n(x) &\leq (r/K_I)w_n(x)u_n(x) \\ &= (r/K_I)e^{-A_0x}e^{-A\int_0^x u_n(s)ds}u_n(x) \\ &\leq Ce^{-A\|u_n\|_\infty \int_0^x \hat{u}_n(s)ds}\|u_n\|_\infty \hat{u}_n(x). \end{aligned}$$

Since $\hat{u}_n \to \hat{u} > 0$ uniformly in $[0,1]$, there exist $c_1, c_2 > 0$ such that $c_1 \leq \hat{u}_n \leq c_2$ and hence

$$\begin{aligned} g_n &:= Ce^{-A\|u_n\|_\infty \int_0^x \hat{u}_n(s)ds}\|u_n\|_\infty \hat{u}_n(x) \\ &\leq Cc_2e^{-Ac_1\|u_n\|_\infty x}\|u_n\|_\infty. \end{aligned}$$

One easily sees from the above inequality that $g_n \to 0$ uniformly in compact subsets of $(0,1]$. It follows that

$$f_n := f(\min\{\alpha v_n, w_n\})u_n \to 0$$

uniformly in compact subsets of $(0,1]$.

We can now prove that $x_0 = 0$. Otherwise, $x_0 \in (0,1]$ and thus $x_n > x_0/2$ for all large n. Since $v_n(x)$ is increasing in x, and $\alpha v_n(x) < w_n(x)$ in $[0, x_n)$, it follows that

$$f(\min\{\alpha v_n, w_n\})u_n \leq f(\alpha v_n(x_0/2))u_n \leq f(w_n(x_0/2))u_n \text{ in } [0, x_0/2].$$

By our earlier estimates for g_n, we find that

$$f(w_n(x_0/2))u_n \leq Cc_2e^{-Ac_1\|u_n\|_\infty(x_0/2)}\|u_n\|_\infty \to 0.$$

Hence $f_n \to 0$ uniformly in $[0,1]$, which implies, by (2.20), that $\zeta_n \to 0$ in $C^{1,\gamma}([0,1])$. In particular, $\zeta_n \to 0$ uniformly in $[0,1]$; but this leads to a contradiction:

$$v_0 - \zeta_n = v_n \leq w_n/\alpha \leq w_n(x_0/2)/\alpha \to 0 \text{ uniformly in } [x_0/2, x_n].$$

Hence $x_0 = 0$. Therefore we have

$$x_n \to 0 \text{ and } u_n/\|u_n\|_\infty \to \phi_0, \tag{2.21}$$

where

$$\phi_0(x) = \exp\left[-\frac{\sigma}{d_1}\int_0^x C_0(s)ds\right] = e^{-\frac{\sigma}{d_1}x}\left(1+\frac{x}{\delta}\right)^{\frac{\sigma}{d_1}\delta}.$$

Step 4: *The limiting profile of* v_n.

Since $v_n \geq 0$ we have $0 \leq \zeta_n \leq v_0$. Moreover, due to (2.20) and the fact that $f_n \to 0$ uniformly in compact subsets of $(0,1]$, we can use standard elliptic regularity theory and a diagonal process to find a subsequence of $\{\zeta_n\}$, still denoted by ζ_n, such that $\zeta_n \to \zeta$ in $C^1([\epsilon, 1])$ for every $\epsilon \in (0,1)$, and ζ satisfies

$$-d_2\zeta'' = 0 \text{ in } (0,1], \quad \zeta'(1) + \beta\zeta(1) = 0, \quad 0 \leq \zeta \leq v_0.$$

It follows that

$$\zeta(x) = \tau(1+\beta^{-1}-x) \text{ for some } \tau \geq 0 \text{ to be determined below.}$$

On the other hand, ζ_n can be explicitly expressed as

$$\zeta_n(x) = d_2^{-1}(1+\beta^{-1}-x)\int_0^1 f_n(s)ds + d_2^{-1}\int_x^1 (x-s)f_n(s)ds.$$

Since

$$\int_0^1 f_n(s)ds = d_2\beta\zeta_n(1) \in (0, d_2\beta v_0(1)],$$

and $f_n \to 0$ uniformly on any compact subset of $(0,1]$, one easily sees that

$$\int_x^1 (x-s)f_n(s)ds \to 0$$

uniformly for $x \in [0,1]$. Thus

$$\tau = \beta\zeta(1) = \lim_{n\to\infty}\beta\zeta_n(1) = d_2^{-1}\lim_{n\to\infty}\int_0^1 f_n(s)ds = d_2^{-1}\lim_{n\to\infty}\int_0^\epsilon f_n(s)ds \tag{2.22}$$

for any $\epsilon \in (0,1)$.

Since $v_n(x)$ is monotone increasing in x and $v_n''(x) \geq 0$, by an elementary argument we see that the fact that $v_n \to v_0 - \zeta$ in $C^1([\epsilon,1])$ for every $\epsilon \in (0,1)$ implies that $v_n \to v_0 - \zeta$ uniformly in $[0,1]$.

We now show that $\tau > 0$. Suppose $\tau = 0$. Then $\zeta_n \to 0$ and hence $v_n \to v_0$ in $C^1([0,1])$. Therefore, due to

$$w_n(0) = w_0 \text{ and } w_n \to 0 \text{ uniformly in } [\epsilon,1]\ \forall \text{ small } \epsilon > 0,$$

when $w_0 > \alpha v_0$, we have $0 < x_n < 1$ for all large n, and

$$w_n(x_n) = \alpha v_n(x_n) \to \alpha v_0 \text{ as } n \to \infty;$$

if $w_0 < \alpha v_0$, then $x_n = 0$ and $w_n(x_n) = w_0$ for all large n; if $w_0 = \alpha v_0$, then either $0 < x_n < 1$ and $w_n(x_n) = \alpha v_n(x_n)$, or $x_n = 0$ and $w_n(x_n) = w_0$; in either case we can conclude that $w_n(x_n) \to w_0$ as $n \to \infty$.

Summarizing, we find that we always have

$$w_0 e^{-A_0 x_n} e^{-A\int_0^{x_n} u_n(s)ds} = w_n(x_n) \to \sigma_0 := \min\{w_0, \alpha v_0\}.$$

Since $x_n \to 0$ and $\hat{u}_n \to \phi_0$, we have $e^{-A_0 x_n} = 1 + o(1)$, and

$$\int_0^{x_n} u_n(s)ds = \|u_n\|_\infty \int_0^{x_n} \hat{u}_n(s)ds = \|u_n\|_\infty x_n \phi_0(0)[1+o(1)] = \|u_n\|_\infty x_n[1+o(1)].$$

Here $o(1)$ denotes a generic sequence converging to 0 as $n \to \infty$. This implies that

$$w_0 e^{-A\|u_n\|_\infty x_n} \to \sigma_0.$$

Hence

$$\|u_n\|_\infty x_n \to \tau_0 := A^{-1} \ln\left(\frac{w_0}{\sigma_0}\right). \tag{2.23}$$

Since we now assume that $\tau = 0$, by (2.22) we must have $\lim_{n\to\infty} \int_0^\epsilon f_n(x)dx = 0$. On the other hand, making use of $x_n \to 0$, $\|u_n\|_\infty x_n \to \tau_0$, $\hat{u}_n \to \phi_0$, and $\phi_0(0) = 1$, we have, for all large n and small ϵ,

$$\begin{aligned}
\int_0^\epsilon f_n(x)dx &\geq \int_{x_n}^\epsilon f(w_n(x))u_n(x)dx \\
&\geq \int_{x_n}^\epsilon f\Big(w_0 e^{-A\epsilon} \exp[-A\|u_n\|_\infty 2\phi_0(0)x]\Big)\frac{\phi_0(0)}{2}\|u_n\|_\infty dx \\
&= \int_{\|u_n\|_\infty x_n}^{\|u_n\|_\infty \epsilon} f\Big(w_0 e^{-A\epsilon} e^{-2Ay}\Big)\left(\frac{1}{2}\right)dy \\
&\to \left(\frac{1}{2}\right)\int_{\tau_0}^\infty f\Big(w_0 e^{-A\epsilon} e^{-2Ay}\Big)dy > 0.
\end{aligned}$$

This contradiction shows that we must have $\tau > 0$.

In order to find the asymptotic limit of the entire sequence $\{v_n\}$, we need to determine the value of τ. Recall that by passing to a subsequence, $v_n \to v_0 - \zeta$ uniformly in $[0, 1]$. Since $x_n \to 0$, we have either

$$w_n(x_n) = \alpha v_n(x_n) \to \alpha[v_0 - \zeta(0)] = \alpha[v_0 - \tau(1 + \beta^{-1})] := \xi_\tau,$$

which is the case when $\xi_\tau < w_0$, or $w_n(x_n) \to w_0$ when $\xi_\tau \geq w_0$. By the expression of $w_n(x_n)$, as before, we deduce that

$$w_0 e^{-A\|u_n\|_\infty x_n} \to \tilde{\xi}_\tau := \min\{w_0, \xi_\tau\}, \quad \|u_n\|_\infty x_n \to \sigma_\tau := A^{-1} \ln\big(w_0/\tilde{\xi}_\tau\big). \tag{2.24}$$

For fixed $\epsilon \in (0, 1)$, we have

$$\int_0^\epsilon f_n(x)dx = \int_0^{x_n} f_n(x)dx + \int_{x_n}^\epsilon f(w_n(x))u_n(x)dx.$$

It is easy to check that, in every possible case, we have

$$\int_0^{x_n} f_n(x)dx = f(\tilde{\xi}_\tau)\phi_0(0)[1 + o(1)]\|u_n\|_\infty x_n = f(\tilde{\xi}_\tau)\sigma_\tau[1 + o(1)].$$

Since we already know that $f(w_n(x))u_n(x) \to 0$ uniformly on any compact subset of $(0, 1]$, we find that, for any fixed $\epsilon_1 \in (0, \epsilon)$ and all large n,

(2.25)
$$
\begin{aligned}
&\int_{x_n}^{\epsilon} f\big(w_n(x)\big)u_n(x)dx\\
&=\int_{x_n}^{\epsilon_1} f\big(w_n(x)\big)u_n(x)dx+o_n(1)\\
&=\int_{x_n}^{\epsilon_1} f\big(w_0e^{-A_0x}e^{-A\int_0^x u_n(s)ds}\big)u_n(x)dx+o_n(1)\\
&=\int_{x_n}^{\epsilon_1} f\big(w_0[1+o_{\epsilon_1}(1)]e^{-A\|u_n\|_\infty \hat{u}_n(0)x[1+o_{\epsilon_1}(1)]}\big)\|u_n\|_\infty\hat{u}_n(0)[1+o_{\epsilon_1}(1)]dx+o_n(1)\\
&=[1+o_{\epsilon_1}(1)]\int_{x_n}^{\epsilon_1} f\big(w_0e^{-A\|u_n\|_\infty\hat{u}_n(0)x}\big)\|u_n\|_\infty\hat{u}_n(0)dx+o_n(1)\\
&=[1+o_{\epsilon_1}(1)]\int_{\|u_n\|_\infty\hat{u}_n(0)x_n}^{\epsilon_1\|u_n\|_\infty\hat{u}_n(0)} f\big(w_0e^{-Ay}\big)dy+o_n(1)\\
&=[1+o_{\epsilon_1}(1)]\bigg[\int_{\sigma_\tau}^{\infty} f\big(w_0e^{-Ay}\big)dy+o_n(1)\bigg]+o_n(1),
\end{aligned}
$$

where $o_n(1)\to 0$ as $n\to\infty$ for fixed ϵ_1, and $o_{\epsilon_1}(1)\to 0$ as $\epsilon_1\to 0$ uniformly in n. For arbitrary $\epsilon_1\in(0,\epsilon)$, we first let $n\to\infty$ and then let $\epsilon_1\to 0$, and we obtain from (2.25) that

$$\lim_{n\to\infty}\int_{x_n}^{\epsilon} f\big(w_n(x)\big)u_n(x)dx=\int_{\sigma_\tau}^{\infty} f\big(w_0e^{-Ay}\big)dy.$$

Therefore

$$\lim_{n\to\infty}\int_0^{\epsilon} f_n(x)dx=f(\tilde{\xi}_\tau)\sigma_\tau+\int_{\sigma_\tau}^{\infty} f\big(w_0e^{-Ay}\big)dy. \tag{2.26}$$

Making use of (2.24) and (2.26), we can rewrite (2.22) as

$$d_2\tau=f(w_0e^{-A\sigma_\tau})\sigma_\tau+\int_{\sigma_\tau}^{\infty} f\big(w_0e^{-Ay}\big)dy. \tag{2.27}$$

It can be easily checked that the function

$$F(\theta):=f(w_0e^{-A\theta})\theta+\int_{\theta}^{\infty} f\big(w_0e^{-Ay}\big)dy$$

satisfies $F'(\theta)<0$ and hence it is decreasing in $[0,\infty)$, with

$$F(0)=\int_0^{\infty} f\big(w_0e^{-Ay}\big)dy=\frac{r}{A}\ln\left(\frac{w_0+K_I}{K_I}\right),\quad F(\infty)=0.$$

From the definition of σ_τ, we find that $\tau\to\sigma_\tau$ is nondecreasing, with

$$\sigma_0=A^{-1}\ln\left(\frac{w_0}{\min\{w_0,\alpha v_0\}}\right),\quad \sigma_{v_0/(1+\beta^{-1})}=\infty.$$

Therefore

$\tau\to F(\sigma_\tau)$ is nonincreasing in $[0,v_0/(1+\beta^{-1}))$ with $F(\sigma_0)>0, F(\sigma_{v_0/(1+\beta^{-1})})=0.$

This implies that (2.27) has a unique solution $\tau \in \big(0, v_0/(1+\beta^{-1})\big)$. Thus,

$$v_n \to v_0 - \tau(1+\beta^{-1} - x) \text{ uniformly in } [0,1],$$

with τ uniquely determined by (2.27). Since $\tau > 0$ is uniquely determined, the above convergence is true for the entire original sequence $\{v_n\}$.

The proof is complete. $\square$

3. The limit of m_* and m^* as $\sigma \to \infty$. In order to investigate the asymptotic behavior of the positive solutions of (1.3) as $\sigma \to \infty$, we need to first understand the limits of m_* and m^* as $\sigma \to \infty$. To stress their dependence on σ, we write $m_* = m_*(\sigma)$ and $m^* = m^*(\sigma)$. Let us recall that (2.6) holds; that is, $m_*(\sigma)$ and $m^*(\sigma)$ are always between the positive numbers $f(\min\{\alpha v_0, w_*(1)\})$ and $f(\min\{\alpha v_0, w_0\})$. We now prove the following result.

THEOREM 3.1.

$$\lim_{\sigma\to\infty} m_*(\sigma) = \lim_{\sigma\to\infty} m^*(\sigma) = f(\min\{\alpha v_0, w_0\}). \tag{3.1}$$

Proof. Since

$$m_*(\sigma) \le m^*(\sigma) \le f(\min\{\alpha v_0, w_0\}),$$

we need only show that

$$\lim_{\sigma\to\infty} m_*(\sigma) = f(\min\{\alpha v_0, w_0\}).$$

Moreover, it suffices to prove this along an arbitrary sequence of positive numbers increasing to ∞. Let $\{\sigma_n\}$ be such a sequence, and denote $m_n = m_*(\sigma_n)$. By definition, there exists $u_n > 0$ in $[0,1]$ such that

$$\begin{cases} -[d_1 u_n' + \sigma_n c^0(x) u_n]' = [f(\min\{\alpha v_0, w_*\}) - m_n]u_n & \text{in } (0,1), \\ d_1 u_n' + \sigma_n c^0(x) u_n = 0 & \text{for } x = 0, 1. \end{cases} \tag{3.2}$$

To simplify the notation, we will write

$$f_0(x) = f(\min\{\alpha v_0, w_*(x)\}).$$

Moreover, we define x_0^* by

$$c^0(x) = C_{x_0^*}(x) = \frac{x - x_0^*}{\delta + |x - x_0^*|}.$$

If $x_0^* = 1$, i.e., $\alpha v_0 \le w_*(x)$ in $[0,1]$ and hence $f_0(x) \equiv f(\alpha v_0)$, then clearly

$$m_n \equiv f(\alpha v_0) = f(\min\{\alpha v_0, w_0\}).$$

So (3.1) holds trivially in this case.

Suppose from now on that $x_0^* \in [0,1)$. Therefore $f_0(x)$ is a constant in $[0, x_0^*)$ and is decreasing in $[x_0^*, 1]$. As before, integrating the first equation of (3.2) we find that $(f_0(x) - m_n)$ must change sign in $(0,1)$, and therefore there exists a unique $x_n \in (0,1)$ such that $f_0(x) > m_n$ in $[0, x_n)$, and $f_0(x) < m_n$ in $(x_n, 1]$.

By (2.6), $\{m_n\}$ is a bounded sequence, and, by passing to a subsequence, we may assume that

$$m_n \to m_0 \le f(\min\{(\alpha v_0, w_0\}) = f_0(x_0^*).$$

To determine the value of m_0, we use several steps.

Step 1: *Change of variables.*

Let

$$\phi_n(x) := \exp\left[-\frac{\sigma_n}{d_1} \int_{x_0^*}^{x} c^0(x)dx \right].$$

Clearly

$$d_1\phi_n' + \sigma_n c^0 \phi_n = 0, \quad \phi_n(x_0^*) = 1, \quad 0 < \phi_n(x) \le 1 \text{ in } [0,1],$$

and $\phi_n \to 0$ uniformly on compact subsets of $[0,1] \setminus \{x_0^*\}$.

Define

$$\psi_n(x) = u_n(x)/\phi_n(x).$$

Then (3.2) becomes

$$\begin{cases} -(d_1\phi_n\psi_n')' = [f_0(x) - m_n]\phi_n\psi_n & \text{in } (0,1), \\ \psi_n' = 0 & \text{for } x = 0,1. \end{cases} \tag{3.3}$$

Define

$$\xi_n(x) = \sigma_n^{-1/2} \int_0^x \frac{1}{\phi_n(s)} ds.$$

Then ξ_n is an increasing function in $[0,1]$, with $\xi_n(0) = 0$ and

$$\xi_n(1) = y_n := \sigma_n^{-1/2} \int_0^1 \frac{1}{\phi_n(x)} dx \to \infty \text{ as } n \to \infty.$$

Let $\eta_n : [0, y_n] \to [0,1]$ be the inverse function of $\xi_n(x)$, and define

$$U_n(y) = \psi_n(\eta_n(y)) = \psi_n(x).$$

From (3.3) a simple calculation shows that

$$\begin{cases} -U_n'' = d_1^{-1}\sigma_n\phi_n^2(\eta_n(y))[f_0(\eta_n(y)) - m_n]U_n & \text{in } (0, y_n), \\ U_n'(0) = U_n'(y_n) = 0. \end{cases} \tag{3.4}$$

Step 2: *Estimates of* $\sigma_n\phi_n^2(\eta_n(y))$.

For our later estimates, we need to analyze the function $\tilde{\phi}_n(y) := \sigma_n\phi_n^2(\eta_n(y))$. To this end, for some $\tau > 0$ small to be determined later, we define $\hat{y}_n$ and Δ_n by

$$\hat{y}_n := \xi_n(x_0^*), \quad \sigma_n^{-\tau} := \sigma_n^{-1/2} \int_{x_0^*}^{x_0^*+\Delta_n} \frac{1}{\phi_n(x)} dx,$$

so that

$$\hat{y}_n \pm \sigma_n^{-\tau} = \xi_n(x_0^* \pm \Delta_n), \quad x_0^* \pm \Delta_n = \eta_n(\hat{y}_n \pm \sigma_n^{-\tau}).$$

We will show that $\tilde{\phi}_n(y)$ behaves like a δ-function concentrating at $y = \hat{y}_n$. For definiteness, we assume that $x_0^* > 0$; the case $x_0^* = 0$ can be treated by a simple modification of the arguments below. Then it is easily seen that as $n \to \infty$,

$$\hat{y}_n \to \infty, \quad y_n - \hat{y}_n \to \infty, \quad \Delta_n \to 0.$$

Using $c^0(x) = \frac{x-x_0^*}{\delta+|x-x_0^*|}$, we can easily check that, for any given small $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ small so that, when $|x - x_0^*| \le \delta_0$,

$$\exp\left[-\frac{\sigma_n}{2\delta d_1}(x-x_0^*)^2\right] \le \phi_n(x) \le \exp\left[-\frac{\sigma_n(1-\epsilon)}{2\delta d_1}(x-x_0^*)^2\right]. \tag{3.5}$$

Therefore, for all large n,

$$\begin{aligned}
\sigma_n^{1/2-\tau} &= \int_{x_0^*}^{x_0^*+\Delta_n} \frac{1}{\phi_n(x)}dx \\
&\le \int_{x_0^*}^{x_0^*+\Delta_n} \exp\left[\frac{\sigma_n}{2\delta d_1}(x-x_0^*)^2\right]dx \\
&= \int_0^{\Delta_n} \exp\left(\frac{\sigma_n}{2\delta d_1}x^2\right)dx \\
&\le \Delta_n \exp\left(\frac{\sigma_n\Delta_n^2}{2\delta d_1}\right),
\end{aligned}$$

$$\begin{aligned}
\sigma_n^{1/2-\tau} &= \int_{x_0^*}^{x_0^*+\Delta_n} \frac{1}{\phi_n(x)}dx \\
&\ge \int_{x_0^*}^{x_0^*+\Delta_n} \exp\left[\frac{\sigma_n(1-\epsilon)}{2\delta d_1}(x-x_0^*)^2\right]dx \\
&= \int_0^{\Delta_n} \exp\left[\frac{\sigma_n(1-\epsilon)}{2\delta d_1}x^2\right]dx \\
&\ge \int_{(1-\epsilon)\Delta_n}^{\Delta_n} \exp\left[\frac{\sigma_n(1-\epsilon)}{2\delta d_1}x^2\right]dx \\
&\ge \epsilon\Delta_n \exp\left[\frac{\sigma_n\Delta_n^2(1-\epsilon)^3}{2\delta d_1}\right].
\end{aligned}$$

It follows that $\sigma_n\Delta_n^2 \to \infty$, and

$$\exp\left(-\frac{\sigma_n\Delta_n^2}{2\delta d_1}\right) \le \Delta_n\sigma_n^{\tau-1/2}, \tag{3.6}$$

and

$$\sigma_n^{1/2}\Delta_n \exp\left[\frac{\sigma_n\Delta_n^2(1-\epsilon)^3}{2\delta d_1}\right] \le \epsilon^{-1}\sigma_n^{1-\tau},$$

which gives

$$\left(\frac{1}{2}\right)\ln(\sigma_n\Delta_n^2) + \frac{(1-\epsilon)^3}{2\delta d_1}\sigma_n\Delta_n^2 \le \ln\left(\epsilon^{-1}\sigma_n^{1-\tau}\right).$$

Since $\ln(\sigma_n\Delta_n^2)=o(\sigma_n\Delta_n^2)$, and

$$\ln(\epsilon^{-1}\sigma_n^{1-\tau})=(1-\tau)\ln\sigma_n+o(\ln\sigma_n),$$

the last inequality above implies that

$$\sigma_n\Delta_n^2\le C_\epsilon\ln\sigma_n \tag{3.7}$$

for some $C_\epsilon>0$ and all large n.

As a consequence of (3.5), (3.6), and (3.7) we have

$$\begin{aligned}\tilde\phi_n(\hat y_n\pm\sigma_n^{-\tau}) &= \sigma_n\phi_n^2(x_0^*\pm\Delta_n)\\ &\le \sigma_n\exp\left[-\frac{\sigma_n(1-\epsilon)}{\delta d_1}\Delta_n^2\right]\\ &\le \sigma_n\left[\Delta_n^2\sigma_n^{2\tau-1}\right]^{(1-\epsilon)}\\ &\le \left[C_\epsilon\ln\sigma_n\right]^{(1-\epsilon)}\sigma_n^{1+2(\tau-1)(1-\epsilon)}\\ &\to 0 \text{ as } n\to\infty,\end{aligned}$$

provided that τ is chosen in the interval $(0,1/2)$ and $\epsilon>0$ is small enough.

From the property of $\phi_n(x)$, we see that the above estimates imply that

$$\tilde\phi_n(y)\to 0 \text{ uniformly in } [0,y_n]\setminus[\hat y_n-\sigma_n^{-\tau},\hat y_n+\sigma_n^{-\tau}], \tag{3.8}$$

and for any $M>0$,

$$\begin{aligned}\int_{\hat y_n-M}^{\hat y_n+M}\tilde\phi_n(y)dy &= \int_{\hat y_n-\sigma_n^{-\tau}}^{\hat y_n+\sigma_n^{-\tau}}\tilde\phi_n(y)dy+o(1)\\ &= \int_{\eta_n(\hat y_n-\sigma_n^{-\tau})}^{\eta_n(\hat y_n+\sigma_n^{-\tau})}\sigma_n\phi_n^2(x)\xi_n'(x)dx+o(1)\\ &= \int_{x_0^*-\Delta_n}^{x_0^*+\Delta_n}\sigma_n^{1/2}\phi_n(x)dx+o(1)\\ &= 2\sigma_n^{1/2}\int_0^{\Delta_n}\exp\left(-\frac{\sigma_n x^2}{2\delta d_1}[1+o(1)]\right)dx+o(1)\\ &= [2+o(1)]\int_0^{\sigma_n^{1/2}\Delta_n}\exp\left(-\frac{x^2}{2\delta d_1}\right)dx+o(1)\\ &= 2\int_0^\infty\exp\left(-\frac{x^2}{2\delta d_1}\right)dx+o(1).\end{aligned}$$

In other words, for any $M>0$,

$$\lim_{n\to\infty}\int_{\hat y_n-M}^{\hat y_n+M}\tilde\phi_n(y)dy=c_0:=2\int_0^\infty\exp\left(-\frac{x^2}{2\delta d_1}\right)dx. \tag{3.9}$$

Step 3: *The limiting profile of* $U_n(y)$.

We now define

$$\begin{aligned}\hat U_n(y) &= U_n(y+\hat y_n-1)/U_n(\hat y_n-1),\\ \hat f_n &= d_1^{-1}\tilde\phi_n(y+\hat y_n-1)\big[f_0\big(\eta_n(y+\hat y_n-1)\big)-m_n\big].\end{aligned}$$

Clearly $\hat{U}_n(0) = 1$, and by (3.4) we have

$$\begin{cases} -\hat{U}_n'' = \hat{f}_n \hat{U}_n \text{ in } (1-\hat{y}_n, y_n + 1 - \hat{y}_n), \\ \hat{U}_n'(1-\hat{y}_n) = \hat{U}_n'(y_n + 1 - \hat{y}_n) = 0. \end{cases} \tag{3.10}$$

From (3.8) we see that

$$\hat{f}_n(y) \to 0 \text{ uniformly in } [1-\hat{y}_n, 1-\sigma_n^{-\tau}] \cup [1+\sigma_n^{-\tau}, y_n + 1 - \hat{y}_n].$$

Since $\hat{U}_n(0) = 1$, the boundedness of $\hat{f}_n$ over $[1-\hat{y}_n, 1-\sigma_n^{-\tau}]$ allows us to apply the Harnack inequality to conclude that $\hat{U}_n$ has a bound C_J independent of n over any bounded interval $J \subset [1-\hat{y}_n, 1-\sigma_n^{-\tau}]$ with $0 \in J$. We can now apply to (3.10) the L^p theory, the Sobolev imbedding theorem, and a standard diagonal argument, to obtain a subsequence of $\{\hat{U}_n\}$, still denoted by $\hat{U}_n$, such that $\hat{U}_n \to \hat{U}$ in $C^1(J)$ for any bounded interval $J \subset (-\infty, 1)$, and $\hat{U}$ satisfies

$$\hat{U}'' = 0 \text{ in } (-\infty, 1), \quad \hat{U}(0) = 1. \tag{3.11}$$

Since $\hat{U}$ is nonnegative in $(-\infty, 1)$, we deduce from (3.11) that

$$\hat{U}(y) = 1 + ay, \quad a \in [-1, 0].$$

Now consider the sequence $\{\hat{U}_n(2)\}$. We claim that this is a bounded sequence. Indeed, from our earlier observation for the sign of $[f_0(x) - m_n]$, we know that the right-hand side of the first equation in (3.4) changes sign from positive to negative when y increases across $\tilde{y}_n := \xi_n(x_n)$. It follows that $U_n''(y)$ changes sign from negative to positive as y increases across $\tilde{y}_n$. Since $U_n'(0) = U_n'(y_n) = 0$, we find that $U_n' \le 0$ in $[0, y_n]$ and hence $U_n(y)$ is nonincreasing in y, which implies that $\hat{U}_n(y)$ is nonincreasing in y and hence $0 \le \hat{U}_n(2) \le \hat{U}_n(0) = 1$. We can now use the fact that $\hat{f}_n \to 0$ uniformly in $[1+\sigma_n^{-\tau}, y_n+1-\hat{y}_n]$, as above, to conclude that, subject to passing to a further subsequence, $\hat{U}_n \to \hat{U}_*$ in $C^1(J)$ for any bounded interval $J \subset (1, \infty)$, and $\hat{U}_*$ satisfies

$$\hat{U}_*'' = 0, \quad 0 \le \hat{U}_* \le 1 \text{ in } (1, \infty).$$

Therefore $\hat{U}_*$ must be a constant, say $\hat{U}_* \equiv b$.

Using (3.9) we find that $\hat{f}_n$ is a bounded sequence in $L^1([0,2])$. By (3.10), we have

$$\hat{U}_n'(y) = \hat{U}_n'(0) - \int_0^y \hat{f}_n(y)\hat{U}_n(y)dy \quad \forall y \in [0,2].$$

Since $\hat{U}_n'(0) \to \hat{U}'(0) = a$ and $0 \le \hat{U}_n(y) \le \hat{U}_n(0) = 1$, the above identity implies that $|\hat{U}_n'(y)| \le C$ for some $C > 0$ and all $n \ge 1$ and $y \in [0,2]$. Therefore $\{\hat{U}_n(y)\}$ is equicontinuous in $[0,2]$. It follows that $\hat{U}_*$ must be a continuous extension of $\hat{U}$. Therefore $b = 1 + a$.

Step 4: *We show that* $m_0 = f(x_0^*)$.

We are now ready to determine the value of m_0 by using our estimates for $\tilde{\phi}_n$ and $\hat{U}_n$. We note that (3.8) and (3.9) imply that, for large n, $\tilde{\phi}_n(y + \hat{y}_n - 1)$ behaves like the δ-function concentrating at $y = 1$. We now use these properties of $\tilde{\phi}_n$ and (3.10) to obtain

$$\hat{U}_n'(2) = \hat{U}_n'(0) - \int_0^2 \hat{f}_n(y)\hat{U}_n(y)dy \to a - [f_0(x_0^*) - m_0](1+a)c_0.$$

But since $\hat{U}_*$ is a constant function over $(1, \infty)$, we have $\hat{U}'_n(2) \to \hat{U}'_*(2) = 0$. Therefore

$$a = [f_0(x_0^*) - m_0](1 + a)c_0.$$

The right-hand side of the above identity is nonnegative, but $a \leq 0$. Therefore we must have $a = 0$ and $m_0 = f_0(x_0^*)$. This implies that the entire original sequence $\{m_n\}$ converges to $f_0(x_0^*)$. Hence (3.1) holds, and the proof is complete. ∎

If we fix m such that $0 < m < f(\min\{\alpha v_0, w_0\})$ and let σ_n be an increasing sequence of positive numbers converging to ∞, then by Theorems 2.3 and 3.1, for all large n, (1.3) with $\sigma = \sigma_n$ has at least one positive solution. Suppose that (u_n, v_n) is such a solution. We will analyze the behavior of (u_n, v_n) as $n \to \infty$. This will be done in Part II; see [DH].

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