In some demographic model individual survives in the last age class

$$L = \begin{pmatrix} m_1 & m_2 & \cdots & \cdots & m_n \\ p_1 & 0 & \cdots & \cdots & 0 \\ 0 & p_2 & \cdots & \cdots & 0 \\ 0 & \cdots & & p_{n-1} & p_n \end{pmatrix}$$

<u>Perron-Frobenius Theorem</u>:

Let A be an $n \times n$ matrix

$\mu(A)$ = spectral radius of $A = \max\{|\lambda|, \lambda$ is an eigenvalue of $A\}$

$A = (a_{ij})$ is nonnegative $\Longleftrightarrow a_{ij} \geq 0$

<u>Theorem</u>: If $n \times n$ matrix A is nonnegative, then

(i) $\mu(A)$ is an eigenvalue

(ii) there is a nonnegative eigenvector v associated with $\mu(A)$

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[LT] Lancaster & Tismenetsky (1985): The theory of matrices

<u>Theorem</u> ([LT], p.536)

If $n\times n$ matrix A is nonnegative and irreducible

then

(i) the matrix A has a positive eigenvalue

$r = \mu(A) > 0$

(ii) there is a positive eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $v_i > 0$

$i = 1, 2 \cdots n$. associated with r

(iii) The eigenvalue r has algebraic multiplicity 1

i.e. $\det(\lambda I - A) = (\lambda - r)^{q_1} \cdots (\lambda - \lambda_n)$, $r = \lambda_1$

$q_1 = 1$

(iv) any nonnegative eigenvector of A is a positive multiple of $\vec{v}$

(v) If $B \geq A$, $B \neq A$ then $\mu(B) > \mu(A)$

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<u>Remark</u>: A matrix A is said to be irreducible if it cannot be put into the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ by reordering $(x_1 \cdots x_n)$.

Q: How to check A is irreducible

Let $P_1 \cdots P_n$ be n vertices.

$a_{ij} \neq 0 \iff$ draw a directed line segment P_iP_j connecting P_i to P_j.

The resulting graph is said to be strongly connected if for each pair (P_i, P_j), there is a directed path $P_iP_{k_1}, P_{k_1}P_{k_2} \cdots P_{k_{r-1}}P_j$.

A ~~seq~~ square matrix is irreducible $\iff$ its directed graph is strongly connected

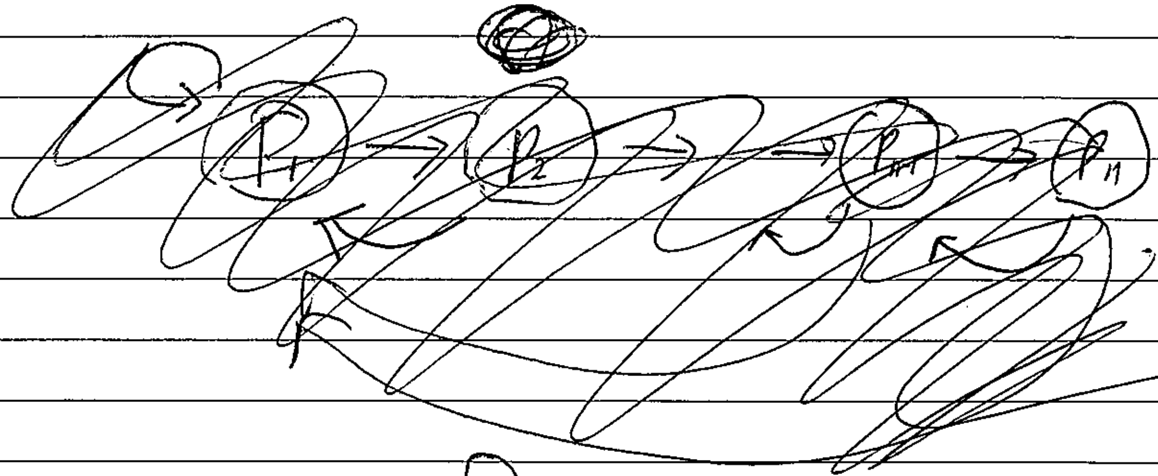
[LT, p.529]

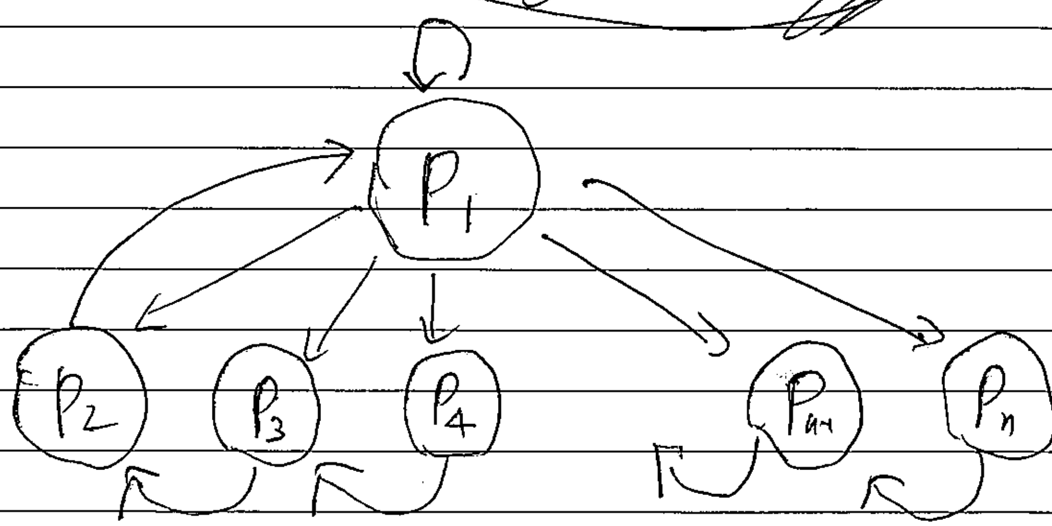
Example:

$$L = \begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ & & \ddots & \\ & & p_{n-1} & 0 \end{pmatrix}$$

$m_i \geq 0$, $p_j > 0$

$i = 1, 2 \cdots n$, $j = 1, 2 \cdots n-1$

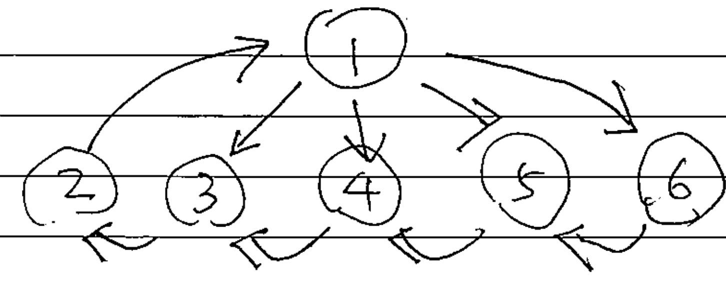




$m_6 = 0$

$n=5$, $P_1, P_2, P_3, P_4 > 0$, $m_1 = 0$, $m_2 = 0$, $m_3 > 0$, $m_4 > 0$, $m_5 > 0$

$P_5 > 0$



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$\vec{x}_{k+1} = L\vec{x}_k \implies \vec{x}_k = L^k \vec{x}_0$

Let $\lambda_1 = r(L)$ be the spectral radius

$\lambda_1, \lambda_2 \cdots \lambda_n \in \mathbb{C}$ be eigenvalues of L

with associated eigenvector $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n$

$\vec{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix} \quad v_{1i} > 0, \; i = 1, 2 \cdots, n$

Then

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$L^k\vec{x}_0 = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_n\lambda_n^k\vec{v}_n$$

$$= \lambda_1^k\left[c_1\vec{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n\right]$$

$\xrightarrow{k\to\infty} \approx \lambda_1^k c_1\vec{v}_1$ in the direction of

dominant eigenvector

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①

$$L = \begin{pmatrix} m_1 & m_2 & \cdots & & m_n \\ p_1 & 0 & \cdots & & 0 \\ 0 & p_2 & 0 & \cdots & 0 \\ & & & & \\ 0 & \cdots & 0 & p_{n-1} & 0 \end{pmatrix}$$

$$\operatorname{Det}(L - \lambda I) = 0$$

$$\begin{vmatrix} m_1-\lambda & m_2 & \cdots & & m_n \\ p_1 & -\lambda & 0 & \cdots & 0 \\ & p_2 & \ddots & -\lambda & \\ & & & & \\ 0 & \cdots & & p_{n-1} & -\lambda \end{vmatrix} = 0$$

$$= (m_1-\lambda)(-\lambda)^{n-1} - p_1 \begin{vmatrix} m_2 & m_3 & \cdots & m_n \\ p_2 & -\lambda & & \\ & \ddots & \ddots & -\lambda \\ 0 & & p_{n-1} & \end{vmatrix}$$

$$= (m_1-\lambda)(-\lambda)^{n-1} - p_1 m_2 (-\lambda)^{n-2} + p_1 p_2 \begin{vmatrix} m_3 & \cdots & m_n \\ p_3 & -\lambda & \\ & p_{n-1} \ddots & -\lambda \end{vmatrix}$$

$$= (m_1-\lambda)(-\lambda)^{n-1} - m_2 p_1 (-\lambda)^{n-2}$$
$$+ m_3 p_1 p_2 (-\lambda)^{n-3} \cdots + (-1)^{n-1} m_n p_1 p_2 \cdots p_{n-1} = 0$$

Let $l_i = p_1 p_2 \cdots p_{i-1}$ with $l_1 = 1$

$$(-\lambda)^n + m_1(-\lambda)^{n-1} + (-1)^1 m_2 l_2 (-\lambda)^{n-2} + (-1)^2 m_3 l_3 (-\lambda)^{n-3} + \cdots + (-1)^{n-1} m_n l_n = 0$$

$$\Leftrightarrow \quad 1 - m_1 \lambda^{-1} - m_2 l_2 \lambda^{-2} - m_3 l_3 \lambda^{-3} - \cdots - m_n l_n \lambda^{-n} = 0$$

$$1 = \sum_{i=1}^{n} \frac{l_i m_i}{\lambda^i}$$ Solve eigenvalue λ by solving polynomials

use MATLAB

The right eigenvector of Lesile Matrix

$L\tilde{u} = \lambda \tilde{u}$ $\quad$ $\tilde{u}$ = dorminant eigenvector

λ = spectral radius

$$\begin{cases} m_1 \tilde{u}_1 + m_2 \tilde{u}_2 + \cdots + m_n \tilde{u}_n = \lambda \tilde{u}_1 \\ P_1 \tilde{u}_1 = \lambda \tilde{u}_2 \\ P_2 \tilde{u}_2 = \lambda \tilde{u}_3 \\ P_{n-1} \tilde{u}_{n-1} = \lambda \tilde{u}_n \end{cases}$$

$$\tilde{u}_i = \frac{\tilde{u}_{i-1} P_{i-1}}{\lambda} \quad i = 2, \ldots n$$

$$\Rightarrow \underline{\tilde{u}_i = \tilde{u}_1 l_i \lambda^{-(i-1)}} \quad (10.4.3)$$

gives all the elements of stable age distribution relative to age 1

Scale form of right eigenvector

$$u_j = \frac{\boxed{l_j \lambda^{-(j-1)}}}{\sum_{i=1}^{n} l_i \lambda^{-(i-1)}}$$

Dorminant eigenvector is the stable age distribution

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The reproductive value of each age class (i.e. the elements of the dominant left eigenvector) measure rel to the reproductive value of newborn ν_1

$\tilde{v} L = \lambda \tilde{v}$ Left eigenvector

$$(\tilde{v}_1, \tilde{v}_2, \dots \tilde{v}_n) \begin{pmatrix} m_1 & m_2 & \cdots & m_{n-1} & m_n \\ p_1 & 0 & \cdots & & 0 \\ 0 & p_2 & \cdots & & 0 \\ & & & & \\ 0 & \cdots & \cdot & p_{n-1} & 0 \end{pmatrix} = \lambda (\tilde{v}_1 \cdots \tilde{v}_n)$$

$$\begin{cases} m_1 \tilde{v}_1 + p_1 \tilde{v}_2 = \lambda \tilde{v}_1 \\ m_2 \tilde{v}_1 + p_2 \tilde{v}_3 = \lambda \tilde{v}_2 \\ \vdots \\ m_{n-1} \cancel{m_n} \tilde{v}_1 + p_{n-1} \tilde{v}_n = \lambda \tilde{v}_{n-1} \Rightarrow m_{n-1} \tilde{v}_1 + p_{n-1} \frac{m_n}{\lambda} \tilde{v}_1 = \lambda \tilde{v}_{n-1} \\ m_n \tilde{v}_1 = \lambda \tilde{v}_n \end{cases}$$

$\boxed{l_i = p_1 \cdots p_{i-1}}$

$$\frac{\tilde{v}_n}{\tilde{v}_1} = \frac{m_n}{\lambda}, \quad \frac{\tilde{v}_{n-1}}{\tilde{v}_1} = \frac{m_{n-1} + \frac{m_n}{\lambda} p_{n-1}}{\lambda} = \frac{m_{n-1}\lambda + m_n p_{n-1}}{\lambda^2}$$

$$\cancel{\frac{\tilde{v}_{n-2}}{\tilde{v}_1}} = m_{n-2} \tilde{v}_1 + p_{n-2} \tilde{v}_{n-1} = \lambda \tilde{v}_{n-2}$$

$$m_{n-2} \tilde{v}_1 + p_{n-2} \cdot \frac{m_{n-1}\lambda + m_n p_{n-1}}{\lambda^2} v_1 = \lambda \tilde{v}_{n-2}$$

$$\Rightarrow \frac{\tilde{v}_{n-2}}{\tilde{v}_1} = \frac{1}{\lambda} \left[m_{n-2} + \frac{p_{n-2} (m_{n-1}\lambda + m_n p_{n-1})}{\lambda^2} \right]$$

$$= \frac{1}{\lambda^3} \left[m_{n-2} \lambda^2 + p_{n-2} m_{n-1} \lambda + m_n p_{n-1} p_{n-2} \right].$$

$$\Rightarrow \frac{\tilde{v}_k}{\tilde{v}_1} = \frac{\lambda_1^{k-1}}{l_k} \sum_{i=k}^{n} \frac{l_i m_i}{\lambda_1^i}$$

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$$\frac{d\lambda}{dp_i} = \frac{u_i V_{i+1}}{E[V_k]}$$

$$E[V_k] = \vec{V}^T \vec{u} = u_1 V_1 + \cdots + u_n V_n$$

$$\frac{d\lambda}{dm_i} = \frac{u_i V_1}{E[V_k]}$$

Exercises: Read chapter 10.

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