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A Remark on the Period of the Periodic Solution in the Lotka–Volterra System*

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An important consideration in the nonlinear predator–prey problem of Lotka–Volterra type is the determination of the period. This paper gives a general expression for the period in terms of the given parameters in the Lotka–Volterra system. We also discuss the qualitative behavior of the period related to the energy level of the Lotka–Volterra system.

1. INTRODUCTION

In this paper we shall give a general expression for the period of the periodic solutions of the following Lotka–Volterra predator–prey system:

$$\frac{dx}{dt} = x(\alpha - \beta y), \tag{1.1}$$

$$\frac{dy}{dt} = y(\delta x - \gamma),$$

$$x(0) = x_0 > 0, \qquad y(0) = y_0 > 0.$$

The Lotka–Volterra equations have been at the root of almost every investigation into population dynamics for the last few decades. They predict the existence of population cycles in which the period is an important parameter. It is well known that (1.1) has a one-parameter family of periodic solutions with the equilibrium $(\gamma/\delta, \alpha/\beta)$ as center point. Volterra [3] computed that for small disturbance of the equilibrium the period of such a solution is $T \approx 2\pi/\sqrt{\alpha\gamma}$. Grasman and Veling [2] gave an asymptotic formula for the period which holds for large disturbances. Frame [1] expressed the period T as power series. In this paper, we express the period T

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as the sum of two integrals and the practical computation of these integrals must be aided by a numerical integration scheme. From the expression of period T we determine the qualitative behavior of the period T related to the energy level C_0 for the periodic orbit. The conclusion is that as a function of C_0, $T(C_0)$ is a strictly increasing function on $(0, \infty)$ with $\lim_{C_0 \to +\infty} T(C_0) = +\infty$.

2. Scaling

In this brief section we choose appropriate nondimensional variables. Hereafter we will use bars over parameters and independent and dependent variables to signify that they appeared in the original equations (1.1). Parameters and independent and dependent variables without bars will be used for the new nondimensional entities. Where no change is made in a parameter or variable, no bars will appear. Thus we let

$$t = \bar{\gamma}\bar{t}, \qquad x(t) = \frac{\bar{\delta}}{\bar{\gamma}}\bar{x}(\bar{t}), \qquad y(t) = \frac{\bar{\beta}}{\bar{\gamma}}\bar{y}(\bar{t}), \qquad \alpha = \frac{\bar{\alpha}}{\bar{\gamma}},$$

$$x_0 = \frac{\bar{\delta}}{\bar{\gamma}}\bar{x}_0, \qquad y_0 = \frac{\bar{\beta}}{\bar{\gamma}}\bar{y}_0.$$

With this change (1.1) becomes

$$\frac{dx}{dt} = x(\alpha - y),$$
$$\frac{dy}{dt} = y(x - 1), \tag{2.1}$$
$$x(0) = x_0 > 0, \qquad y(0) = y_0 > 0.$$

3. Main Result

In this section we state and prove our main result. First we reduce (2.1) to an equation of Van der Pol type. We denote $\cdot = d/dt$. From (2.1) we have

$$y = \alpha - \frac{\dot{x}}{x} \qquad \text{and} \tag{3.1}$$

$$\ddot{x} = \dot{x}(\alpha - y) - x\dot{y} = \dot{x}(\alpha - y) - xy(x - 1) = \frac{(\dot{x})^2}{x} - x(x - 1)\left(\alpha - \frac{\dot{x}}{x}\right),$$

i.e.,

$$\ddot{x} - \frac{(\dot{x})^2}{x} - \dot{x}(x-1) + \alpha x(x-1) = 0. \tag{3.2}$$

Next we simplify Eq. (3.2) by a transform $x = e^z$ or $z = \ln x$. Then we have $\dot{x} = \dot{z}e^z$, $\ddot{x} = \ddot{z}e^z + (\dot{z})^2 e^z$ and

$$\ddot{z} - \dot{z}(e^z - 1) + \alpha(e^z - 1) = 0. \tag{3.3}$$

Let $w = \dot{z}$. Then we rewrite (3.3) into the following system of ordinary differential equations:

$$\begin{aligned} \dot{z} &= w, \\ \dot{w} &= (w - \alpha)(e^z - 1). \end{aligned} \tag{3.4}$$

We may solve (3.4) by separation of variables, i.e.,

$$\frac{w}{w-\alpha}\, dw = (e^z - 1)\, dz. \tag{3.5}$$

With these transforms we are in a position to compute the period T of the periodic solution of (2.1). First we note that the periodic orbit of (2.1) is determined by the first integral

$$\int_1^x \frac{\xi - 1}{\xi}\, d\xi + \int_\alpha^y \frac{\eta - \alpha}{\eta}\, d\eta = C_0,$$

where

$$C_0 = \int_1^{x_0} \frac{\xi - 1}{\xi}\, d\xi + \int_\alpha^{y_0} \frac{\eta - \alpha}{\eta}\, d\eta,$$

or

$$x - 1 - \ln x + y - \alpha - \alpha \ln \frac{y}{\alpha} = C_0.$$

where

$$C_0 = x_0 - 1 - \ln x_0 + y_0 - \alpha - \alpha \ln \frac{y_0}{\alpha}.$$

Let x_{min}, x_{max}, with $x_{min} < x_{max}$, be the two roots of $x - 1 - \ln x = C_0$, i.e., the points $(x_{\min}, \alpha)$ and $(x_{\max}, \alpha)$ are the two extreme points of the periodic orbit of (2.1) on the left and the right, respectively. Now we compute the "time" T_1 spent by the trajectory of (2.1) travelling from $(x_{\min}, \alpha)$ to $(x_{\max}, \alpha)$. Suppose at $t=0$, $x=x_{\min}$, $y=\alpha$. Since $z=\ln x$, $w=\dot{z}=\dot{x}/x=\alpha-y$ and $0<y\leqslant\alpha$, it follows that $z=\ln(x_{\min})$, $w=0$ as $t=0$ and $\alpha>w\geqslant 0$ on $0\leqslant t\leqslant T_1$. From (3.5) it follows that

$$\int_0^w \frac{\xi}{\xi-\alpha}\,d\xi=\int_{\ln(x_{\min})}^z (e^\eta-1)\,d\eta. \tag{3.6}$$

Set $F(w)=\int_0^w (\xi/(\xi-\alpha))\,d\xi$. Then $F(w)=w+\alpha\ln(|\alpha-w|/\alpha)$. It is easy to see that $F(0)=0$, $F(w)$ is negative on $(-\infty,0)$ and $(0,\alpha)$, strictly decreasing on $(0,\alpha)$, strictly increasing on $(-\infty,0)$ and $\lim_{w\to-\infty}F(w)=\lim_{w\to\alpha}F(w)=-\infty$. Set $G(z)=\int_{\ln(x_{\min})}^z (e^\eta-1)\,d\eta$. Then $G(z)=(e^z-z)-(x_{\min}-\ln(x_{\min}))$ or $G(z)=(e^z-z)-(C_0+1)$. It is easy to see that $G(z)\leqslant 0$ on $[\ln(x_{\min}), \ln(x_{\max})]$ with its minimal value $G(0)=-C_0<0$ and maximal values $G(\ln(x_{\min}))=G(\ln(x_{\max}))=0$.

Let $F_1(w)$ be the restriction of $F(w)$ on $[0,\alpha)$. Then F_1 is $1-1$ on $[0,\alpha)$. We rewrite (3.6) as

$$w=F_1^{-1}(G(z)). \tag{3.7}$$

Since $w=dz/dt$, it follows that

$$T_1=\int_{\ln(x_{\min})}^{\ln(x_{\max})}\frac{dz}{F_1^{-1}(G(z))}. \tag{3.8}$$

We note that the integral in (3.8) is an improper Riemann-integral on $(\ln(x_{\min}), \ln(x_{\max}))$. Since w is closed to 0, $F_1(w)=F_1(0)+F_1'(0)w+F''(0)\,w^2+\cdots$ is closed to $w^2/-\alpha$ and hence $F_1^{-1}(u)$ is close to $\sqrt{-\alpha u}$.

Similarly, we can compute the "time" T_2 spent by the trajectory of (2.1) travelling from $(x_{\max}, \alpha)$ to $(x_{\min}, \alpha)$. Suppose at $t=0$, $x=x_{\max}$, $y=\alpha$. Since $w=\alpha-y$, $y\geqslant\alpha$, it follows that $z=\ln(x_{\max})$, $w=0$ as t $=0$ and $w\leqslant 0$ on $0\leqslant t\leqslant T_2$. Let $F_2(w)$ be the restriction of $F(w)$ on $(-\infty,0]$. Then $F_2$is 1–1 on $(-\infty,0]$. We rewrite (3.6) as $w=F_2^{-1}(G(z))$ and it follows that

$$T_2=\int_{\ln(x_{\max})}^{\ln(x_{\min})}\frac{dz}{F_2^{-1}(G(z))}. \tag{3.9}$$

Similarly, the integral in (3.9) is an improper Riemann-integral. From (3.8), (3.9), we have the following, our main result.

THEOREM 1. *The period T of the periodic solution of* (2.1) *can be expressed as*

$$T=\int_{\ln(x_{\min})}^{\ln(x_{\max})}\left[\frac{1}{F_1^{-1}(G(z))}-\frac{1}{F_2^{-1}(G(z))}\right]dz, \tag{3.10}$$

where $x_{\max}$, $x_{\min}$ are the two roots of $x-1-\ln x=C_0$, $C_0=x_0-1-\ln x_0+y_0-\alpha-\alpha\ln(y_0/\alpha)$, $G(z)=e^z-z-(C_0+1)\leqslant 0$, on $[\ln(x_{\min}), \ln(x_{\max})]$, and $F_1(w)$ and $F_2(w)$ are the restrictions of $F(w)$ on $[0,\alpha)$, $(-\infty,0)$, respectively, $F(w)=w+\alpha\ln((\alpha-w)/\alpha)$.

Remark. In the practical computation of the period T in Theorem 1, first we have to evaluate $x_{\max}$, $x_{\min}$, the roots of $f(x)=x-1-\ln x-C_0=0$. Since the function $f(x)$ is strictly increasing on $[1,\infty)$ and strictly decreasing on $(0,1]$, we can solve $x_{\max}$ and $x_{\min}$ simply by an iteration procedure. Then the next step is to evaluate the improper integrals

$$\int_{\ln(x_{\min})}^{\ln(x_{\max})}\frac{1}{F_1^{-1}(G(z))}\,dz \qquad\text{and}\qquad \int_{\ln(x_{\min})}^{\ln(x_{\max})}\frac{-1}{F_2^{-1}(G(z))}\,dz.$$

We note that $F_1(w)$ is strictly decreasing on $[0,\alpha)$ and $F_2(w)$ is strictly increasing on $(-\infty,0]$. For each z, $\ln(x_{\min})<z<\ln(x_{\max})$, we set $g(z)=F_1^{-1}(G(z))$. Then we have $F_1(g(z))=G(z)$. We can find $g(z)$ by solving $F_1(w)-G(z)=0$. Since $F_1(w)$ is strictly decreasing, by an iteration process we can easily evaluate $g(z)$. Hence by a numerical integration scheme we obtain an approximation of

$$\int_{\ln(x_{\min})}^{\ln(x_{\max})}\frac{1}{F_1^{-1}(G(z))}\,dz.$$

Similarly, we apply the same process to

$$\int_{\ln(x_{\min})}^{\ln(x_{\max})}\frac{-1}{F_1^{-1}(G(z))}\,dz.$$

As we see in the remark, the numerical computation of the period based on (3.10) is possible, but not practical. However, we may apply (3.10) to obtain the qualitative behavior of T related to the energy level C_0 which is determined by initial values x_0, y_0.

First we observe that that function G: $[\ln(x_{\min}), \ln(x_{\max})]\to[0,-C_0]$ is strictly decreasing on $[\ln(x_{\min}),0]$ and is strictly increasing on $[0,\ln(x_{\max})]$. We denote $G_1=G|_{[\ln(x_{\min}),0]}$ and $G_2=G|_{[0,\ln(x_{\max})]}$. Since $x_{\min}$ and $x_{\max}$ are functions of C_0, from (3.10) T is obviously a function of C_0. We denote $T=T(C_0)$.

THEOREM 2. *$T(C_0)$ is a strictly increasing function of C_0 satisfying $\lim_{C_0 \to +\infty} T(C_0) = +\infty$.*

Proof. From (3.10) we write

$$\begin{aligned} T(C_0) &= \int_{\ln(x_{\min})}^{0} \frac{dz}{F_1^{-1}(G_1(z))} + \int_{0}^{\ln(x_{\max})} \frac{dz}{F_1^{-1}(G_2(z))} \\ &\quad + \int_{0}^{\ln(x_{\min})} \frac{dz}{F_2^{-1}(G_1(z))} + \int_{\ln(x_{\max})}^{0} \frac{dz}{F_2^{-1}(G_2(z))} \\ &= I_1(C_0) + I_2(C_0) + I_3(C_0) + I_4(C_0). \end{aligned}$$

For each $i = 1, 2, 3, 4$, $I_i(C_0) > 0$ and we shall show that each $I_i(C_0)$ is a strictly increasing function on $(0, \infty)$. Forst we consider $I_1(C_0)$. Introduce a new variable θ, $0 \leqslant \theta \leqslant \pi/2$, by

$$F_1^{-1}(G_1(z)) = F_1^{-1}(-C_0) \sin \theta. \tag{3.11}$$

Then $G_1(z) = F_1(F_1^{-1}(-C_0) \sin \theta)$ and hence

$$(e^z - 1)\, dz = \frac{F_1^{-1}(-C_0) \sin \theta}{F_1^{-1}(-C_0) \sin \theta - \alpha} \cdot F_1^{-1}(-C_0) \cos \theta \, d\theta,$$

$$z = (G_1^{-1} \circ F_1)(F_1^{-1}(-C_0) \sin \theta), \tag{3.12}$$

$$I_1(C_0) = \int_0^{\pi/2} \left(\frac{1}{e^z - 1}\right) \cdot \left(\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0) \sin \theta - \alpha}\right) \cos \theta \, d\theta \tag{3.13}$$

$$= \int_0^{\pi} f(C_0, \theta) \cos \theta \, d\theta.$$

Since $G_1^{-1} \circ F_1$ is strictly increasing on $[0, F_1^{-1}(-C_0)]$, from (3.12) we have

$$\frac{\partial z}{\partial C_0} = (G_1^{-1} \circ F_1)' \, (F_1^{-1}(-C_0) \sin \theta) \cdot \sin \theta \cdot (F_1^{-1})'(-C_0) \cdot (-1) \tag{3.14$_1$}$$

$$> 0 \qquad \text{on} \quad 0 < \theta < \pi/2.$$

On the other hand, we have

$$\frac{1}{e^z - 1} < 0, \tag{3.15$_1$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{1}{e^z - 1}\right) = \frac{-e^z}{(e^z - 1)^2} \frac{\partial z}{\partial C_0} < 0, \tag{3.16$_1$}$$

$$\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0)\sin\theta-\alpha}<0, \tag{3.17$_1$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0)\sin\theta-\alpha}\right)=\frac{\alpha(F_1^{-1})'(-C_0)}{(F_1^{-1}(-C_0)\sin\theta-\alpha)^2}<0, \tag{3.18$_1$}$$

and hence

$$\frac{\partial}{\partial C_0}f(C_0,\theta)>0 \qquad \text{on} \quad 0<\theta<\pi/2. \tag{3.19$_1$}$$

Let $C_{01}>C_{02}$. Then

$$I_1(C_{01})-I_1(C_{02})=\int_0^{\pi/2}(C_{01}-C_{02})\frac{\partial f}{\partial C_0}(\xi,\theta)\cos\theta\,d\theta>0$$

for some ξ, $C_{02}<\xi<C_{01}$. Hence $I_1(C_0)$ is a strictly increasing function.

For the integral $I_2(C_0)$, we let $F_1^{-1}(G_2(z))=F_1^{-1}(-C_0)\sin\theta$ and have

$$I_2(C_0)=-\int_0^{\pi/2}\frac{1}{e^z-1}\cdot\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0)\sin\theta-\alpha}\cos\theta\,d\theta,$$

$$z=(G_2^{-1}\circ F_1)(F_1^{-1}(-C_0)\sin\theta), \qquad 0\leqslant\theta\leqslant\pi/2.$$

Similarly, we have that for $0<\theta<\pi/2$,

$$\frac{\partial z}{\partial C_0}<0, \tag{3.14$_2$}$$

$$\frac{1}{e^z-1}>0, \tag{3.15$_2$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{1}{e^z-1}\right)>0, \tag{3.16^2}$$

$$\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0)\sin\theta-\alpha}<0, \tag{3.17$_2$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{F_1^{-1}(-C_0)}{F_1^{-1}(-C_0)\sin\theta-\alpha}\right)<0. \tag{3.18$_2$}$$

Hence $I_2(C_0)$ is strictly increasing on $(0,\infty)$.

For the integral $I_3(C_0)$, we let $F_2^{-1}(G_1(z))=F_2^{-1}(-C_0)\sin\theta$ and have

$$I_3(C_0)=-\int_0^{\pi/2}\frac{1}{e^z-1}\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta-\alpha}\cos\theta\,d\theta,$$

$$z=(G_1^{-1}\circ F_2)(F_2^{-1}(-C_0)\sin\theta).$$

We have that for $0 < \theta < \pi/2$,

$$\frac{\partial z}{\partial C_0} > 0, \tag{3.14$_3$}$$

$$\frac{1}{e^z - 1} < 0, \tag{3.15$_3$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{1}{e^z - 1}\right) < 0, \tag{3.16$_3$}$$

$$\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta - \alpha} > 0, \tag{3.17$_3$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta - \alpha}\right) > 0. \tag{3.18$_3$}$$

Hence $I_3(C_0)$ is strictly increasing on $(0, \infty)$.

For the integral $I_4(C_0)$, we let $F_2^{-1}(G_2(z)) = F_2^{-1}(-C_0)\sin\theta$ and have

$$I_4(C_0) = \int_0^{\pi/2} \frac{1}{e^z - 1}\,\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta - \alpha}\cos\theta\, d\theta,$$
$$z = (G_2^{-1}\circ F_2)(F_2^{-1}(-C_0)\sin\theta).$$

Similarly, we have for $0 < \theta < \pi/2$,

$$\frac{\partial z}{\partial C_0} < 0, \tag{3.14$_4$}$$

$$\frac{1}{e^z - 1} > 0, \tag{3.15$_4$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{1}{e^z - 1}\right) > 0, \tag{3.16$_4$}$$

$$\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta - \alpha} > 0, \tag{3.17$_4$}$$

$$\frac{\partial}{\partial C_0}\left(\frac{F_2^{-1}(-C_0)}{F_2^{-1}(-C_0)\sin\theta - \alpha}\right) > 0. \tag{3.18$_4$}$$

Hence $I_4(C_0)$ is strictly increasing on $(0, \infty)$.

To show $T(C_0) \to +\infty$ as $C_0 \to +\infty$, we only need to show

$\lim_{C_0 \to +\infty} I_2(C_0) = +\infty$. Since $1/F_1^{-1}(G_2(z))$ is a strictly increasing function on $(0, \ln(x_{\max}))$, we have

$$I_2(C_0) \geqslant \int_0^{\ln(x_{\max})} \frac{dz}{F_1^{-1}(G_2(0))}$$
$$= \frac{1}{F_1^{-1}(-C_0)} \ln(x_{\max}).$$

As $C_0 \to +\infty$, $F_1^{-1}(-C_0) \to \alpha$ and $\ln(x_{\max}) \to +\infty$. Hence $\lim_{C_0 \to +\infty} I_2(C_0) = +\infty$.

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