A NONLOCAL PROBLEM FROM CONSERVATION BIOLOGY*

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Abstract. In this paper we study a mathematical model arising from crop raiding of large-bodied mammals living in the bio-diversity-rich tropics. The topic is important because it involves highly threaten species that can cause significant economic damage and be killed in retribution. The mathematical model consists of a non-local spatially heterogeneous parabolic problem of logistic type. Our main analytical result characterizes the existence of positive solutions of the model and it provides us with some multiplicity results. The numerics computes them and ascertains their local stability in a simple class of symmetric one-dimensional prototype models.

Key words. Degenerate non-local parabolic problems, Existence and multiplicity of steady state solutions, global bifurcation diagrams, path-following coupled with collocation, local stability of steady-states

AMS subject classifications. 35K10, 35J61, 65N35, 92B05

1. Introduction.

In this paper we study a mathematical model arising from crop raiding of large-bodied mammals living in the biodiversity-rich tropics. The topic is important because it involves highly threaten species that can cause significant economic damage and be killed in retribution. There are several reports of crop raiding exist for chimpanzees *Pan troglodytes* in Uganda and Guinea Republic; gorilla *Gorilla beringei* in Rwanda; orangutans (*Pongo abelii* and *Pango pygmaeus*) in Indonesia and Malaysia. The reader can find the paper Campbell-Smith et al. [19] and the references therein. Resolving human-wildlife conflict is a conservation imperative because these species are among the most threaten on earth; there large body size means that they can cause substantial economic loss to farmers through crop raiding. It is important to find mitigation strategies to prevent this form of human-wildlife conflict. It is our purpose to understand this problem by constructing mathematical models and studying how the wildlife population behaves.

Let Ω be a nice bounded domain with $\bar{\Omega}_0 \subset \Omega$, the forest region; $\Omega \setminus \Omega_0$ representing the region of farm or orchard. Let $U(x,t)$ be the population density of the mammal at position x and time t. We assume that forest is safe for the mammal species, but is of poor resource; the farm is of rich resources, but is dangerous. The basic assumptions are that the mammals cannot survive if they only stay in the forest and do not attempt to go out to search food; they cannot produce offspring in the farm for it is not a safe place for reproduction; the per capital birth rate is proportional to the total population in the farm. Based on these assumptions the model takes the form

$$\begin{cases} U_t = d\Delta U + \gamma V(x) \left(\int_{\Omega\setminus\Omega_0} U(y,t)\, dy \right) U - U^p & \text{in } \Omega, \quad t>0, \\ \frac{\partial U}{\partial n} = 0 & \text{on } \partial\Omega, \quad t>0, \\ U(\cdot,0) = U_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $V(x)$ stands for the characteristic function of Ω_0

$$V(x) := \begin{cases} 1, & \text{if } x \in \Omega_0, \\ 0, & \text{if } x \in \Omega \setminus \Omega_0, \end{cases} \tag{1.2}$$

$\gamma > 0$ is a parameter measuring the birth rate of the species in the forest, $p \geq 2$ takes into account the crowding effects of the population in Ω, $d > 0$ measures the diffusivity of the species in Ω, and n stands for the outward unit normal to Ω along the territory edges $\partial\Omega$. As a previous step to ascertain the dynamics of (1.1) one has to study its non-negative steady states, which are the non-negative solutions of the semi-linear non-local elliptic problem

$$\begin{cases} -d\Delta U = \gamma V(x) \left(\int_{\Omega\setminus\Omega_0} U(y)\, dy \right) U - U^p & \text{in } \Omega, \\ \frac{\partial U}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

As the change of variable

$$u := d^{\frac{-1}{p-1}} U, \qquad \lambda := \gamma\, d^{-\frac{p-2}{p-1}}, \tag{1.4}$$

transforms (1.3) into

$$\begin{cases} -\Delta u = \lambda V(x) \left(\int_{\Omega\setminus\Omega_0} u(y)\, dy \right) u - u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

we will focus our attention in problem (1.5), which exhibits two types of non-negative solutions: the *trivial solution* $u = 0$ and the *positive solutions* $u > 0$, which, according to the strong maximum principle, satisfy $u \gg 0$, in the sense that $u(x) > 0$ for all $x \in \bar{\Omega}$. The main goal of this paper is characterizing the existence of positive solutions of (1.5) and computing them in a simple symmetric one-dimensional prototype model in order to ascertain their local attractive properties. As (1.1) is a non-local parabolic model, our analysis is fraught with a number of technical difficulties inherent to the lack of the standard comparison techniques available for local problems. So, even the problem of ascertaining the local stability of the positive solutions seems very intricate to handle.

As the nonlinearity has a discontinuity along $\partial\Omega_0$, the weak solutions of (1.5) are not classical solutions, but *strong solutions*, i.e. solutions in $\cap_{q>1} W^{2,q}(\Omega)$. Consequently, they must be of class $\mathcal{C}^{1+\nu}(\bar{\Omega})$ for all $\nu < 1$, are twice differentiable almost everywhere in Ω and satisfy (1.5) a.e. in Ω (see, e.g., López-Gómez [15, Th. 4.1.8] and Stein [22, Th. VIII.1]).

Note that $\gamma > 0$ implies $\lambda > 0$ and that, actually, $\lambda > 0$ is necessary for the existence of a positive solution of (1.5), because, if (λ, u) solves (1.5) with $u > 0$, then, $u \gg 0$ and hence,

$$\lambda \int_{\Omega_0} u \int_{\Omega\setminus\Omega_0} u = \int_\Omega u^p > 0. \tag{1.6}$$

The distribution of this paper is the following. Section 2 deals with a one-dimensional symmetric version of (1.5). Our main result there establishes that for every $p \geq 2$ there exists $\lambda_c = \lambda_c(p) \geq 0$ such that (1.5) possesses at least a positive solution for each $\lambda > \lambda_c(p)$, and no positive solution for $\lambda < \lambda_c(p)$ if $\lambda_c(p) > 0$. Section 3 combines a pseudo-spectral method with collocation and a path-following solver to compute the bifurcation diagrams of positive solutions for $p = 2$ and $p = 3$ in a case when $|\Omega_0| > |\Omega \setminus \Omega_0|$. All computed positive solutions have been locally stable, though we found an important difference between the limiting case $p = 2$ and the general case when $p > 2$. Namely, though according to the numerics $\lambda_c(2) > 0$, it seems that $\lambda_c(p) = 0$ for all $p > 2$. Section 4 determines rigorously all the admissible bifurcation points to positive solutions from $(\lambda, 0)$ for $p \geq 2$. Namely, $\lambda = 0$ if $p > 2$ and $\lambda = \lambda^* := |\Omega|/(|\Omega_0||\Omega \setminus \Omega_0|)$ if $p = 2$. This change of behavior is easily explained by the fact that, within the range $0 < \lambda < \lambda^*$ and $p = 2$, the trivial solution $u = 0$ perturbs into a positive solution when p separates away from 2. Section 5 studies the general multidimensional problem (1.5). By introducing a secondary parameter of spectral nature one can use some techniques from global bifurcation theory, based on the existence of a priori bounds in the auxiliary parameter, to prove that in case $p > 2$ (1.5) admits a positive solution if, and only if, $\lambda > 0$. Finally, in Section 6 we show that indeed $(\lambda, u) = (\lambda^*, 0)$ is a bifurcation point from $u = 0$ to a curve of positive solutions of (1.5) if $p = 2$, and prove that the bifurcation is supercritical if $|\Omega_0| > |\Omega \setminus \Omega_0|$ while it is subcritical if $|\Omega_0| < |\Omega \setminus \Omega_0|$. In particular, for the one-dimensional symmetric prototype model, $\lambda_c(2) < \lambda^*$ if $|\Omega_0| < |\Omega \setminus \Omega_0|$ and hence, for every $\lambda \in (\lambda_c(2), \lambda^*)$, the model should have at least two positive solutions, and three for $p > 2$, $p \sim 2$, as zero perturbs into another positive solution of small amplitude, but this particular issue will be dealt with in a forthcoming paper.

The latest multiplicity result is slightly paradoxical from the point of view of the applications, as it establishes that the smaller is the forest region, measured by $|\Omega_0|$, the larger is the complexity of the dynamics of the model, measured by the number of steady states.

2. The one-dimensional problem. Throughout this section, we assume that

$$N = 1, \quad a < b, \quad 0 < L < (b-a)/2, \quad \Omega = (a, b), \quad \Omega_0 = (a + L, b - L), \tag{2.1}$$

and search for positive solutions of

$$\begin{cases} -u'' = 2\lambda V(x) \left(\int_a^{a+L} u(y)\, dy \right) u - u^p & \text{in } (a, (a+b)/2), \\ u'(a) = u'((a+b)/2) = 0. \end{cases} \tag{2.2}$$

By reflection about $(a+b)/2$, these solutions provide us with the symmetric positive solutions of (1.5) for the special choice (2.1). The main result of this section can be stated as follows

THEOREM 2.1. *For every $p \geq 2$ there exists $\lambda_c = \lambda_c(p) \geq 0$ such that* (2.2) *possesses a positive solution for each $\lambda > \lambda_0(p)$, at least, and no positive solution for $\lambda < \lambda_c(p)$ if $\lambda_c(p) > 0$.*

The proof of this result will follow after a series of technical results, which have been distributed in a number of subsections. Throughout this paper, given a second order linear elliptic operator $\mathfrak{L}$ in a nice bounded open set $\mathcal{O}$ and a mixed boundary operator $\mathfrak{B}$ of the general type considered in [2] and [15], we will denote by $\sigma[\mathfrak{L}, \mathcal{O}, \mathfrak{B}]$ the *principal eigenvalue* of the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \tau\varphi & \text{in } \mathcal{O}, \\ \mathfrak{B}\varphi = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

If $\mathfrak{B}\varphi = \partial_n \varphi$, we will set $\mathfrak{B} = \mathfrak{N}$, while we simply write $\mathfrak{B} = \mathfrak{D}$ if $\mathfrak{B}\varphi = \varphi$. The reader is sent to Chapters 8 and 9 of [15] for the proof of the most important properties of $\sigma[\mathfrak{L}, \mathcal{O}, \mathfrak{B}]$.

2.1. An auxiliary uniqueness result. Subsequently, to shorten notations, we denote

$$m := (a+b)/2. \tag{2.3}$$

As $\int_a^{a+L} u > 0$ for any positive solution u of (2.2), the solutions of (2.2) must solve

$$\begin{cases} -u'' = \kappa V(x) u - u^p & \text{in } (a, m), \\ u'(a) = u'(m) = 0, \end{cases} \tag{2.4}$$

for some constant $\kappa > 0$. The next result holds.

LEMMA 2.2. *The problem* (2.4) *admits, at most, one positive solution.*

Proof. Suppose that $u_1 \neq u_2$ are positive solutions of (2.4) and set

$$w := u_1 - u_2.$$

Then,

$$-w'' = \kappa V(x) w - (u_1^p - u_2^p) \qquad \text{in } \ (a,m).$$

On the other hand, for every $x \in (a,m)$, we have that

$$u_1^p(x) - u_2^p(x) = p \int_0^1 [t u_1(x) + (1-t) u_2(x)]^{p-1} \, dt \, (u_1(x) - u_2(x))$$

and hence,

$$\big(-D^2 + I(x) - \kappa V(x)\big)\, w = 0 \quad \text{in } \ (a,m), \tag{2.5}$$

where we have denoted

$$D := d/dx, \qquad I(x) := p \int_0^1 [t u_1(x) + (1-t) u_2(x)]^{p-1} \, dt, \qquad x \in (a,m).$$

As $w'(0) = w'(m) = 0$, by the dominance of the principal eigenvalue in (a,m) (e.g., see [15, Th. 7.9]), we find from (2.5) that

$$\sigma[-D^2 + I(x) - \kappa V(x), (a,m), \mathfrak{N}] \leq 0. \tag{2.6}$$

On the other hand, as $(1-t) u_2(x) > 0$ for all $t \in (0,1)$ and $x \in [a,m]$, we have that

$$I(x) > p \int_0^1 t^{p-1} \, dt \, u_1^{p-1}(x) = u_1^{p-1}(x)$$

for all $x \in [a,m]$. Therefore, by the monotonicity of the principal eigenvalue with respect to the potential, we obtain that

$$\sigma[-D^2 + I(x) - \kappa V(x), (a,m), \mathfrak{N}] > \sigma[-D^2 + u_1^{p-1} - \kappa V(x), (a,m), \mathfrak{N}] = 0,$$

because $u_1'(0) = u_1'(m) = 0$ and

$$(-D^2 + u_1^{p-1} - \kappa V) u_1 = 0 \quad \text{in } \ (a,m).$$

Naturally, according to (2.6), this is impossible. The proof is complete. □

COROLLARY 2.3. *If $u_1 \neq u_2$ solve* (2.2)*, then* $\int_a^{a+L} u_1 \neq \int_a^{a+L} u_2$.

Proof. In the contrary case, both solutions solve (2.4) with

$$\kappa = 2\lambda \int_a^{a+L} u_1(x) \, dx = 2\lambda \int_a^{a+L} u_2(x) \, dx,$$

which is impossible, by Lemma 2.2. □

2.2. The profile of the solutions of (2.2) **in** $(a, a+L)$**.** The solutions of (2.2) in $(a, a+L)$ satisfy the Cauchy problem

$$u'' = u^p, \qquad u(a) = h, \quad u'(a) = 0, \tag{2.7}$$

for some $h > 0$ to be determined. These solutions are explosive for sufficiently large h. Indeed, as soon as the solution u is defined in $[a, x]$, one has that

$$u'(x) = \int_a^x u^p(y)\,dy > 0 \tag{2.8}$$

and hence, $u(x) > h$ and $u'(x) > 0$ for all $x \in (a, T_h)$, where $T_h \in (0, \infty]$ stands for the maximal existence time of the unique solution of (2.7). Thus, multiplying the differential equation by $u'(x)$ and integrating yields

$$u'(x) = \sqrt{\frac{2}{p+1}\left(u^{p+1}(x) - h^{p+1}\right)} \tag{2.9}$$

for all $x \in (a, T_h)$, and therefore,

$$x = a + \int_a^x \frac{u'(y)}{\sqrt{\frac{2}{p+1}\left(u^{p+1}(y) - h^{p+1}\right)}}\,dy = a + \sqrt{\frac{p+1}{2}}\,h^{-\frac{p-1}{2}} \int_1^{u(x)/h} \frac{d\theta}{\sqrt{\theta^{p+1} - 1}}.$$

Thus, letting $x \to T_h$ in the previous identity, it becomes apparent that

$$T_h = a + \sqrt{\frac{p+1}{2}}\,h^{-\frac{p-1}{2}} \int_1^\infty \frac{d\theta}{\sqrt{\theta^{p+1} - 1}} < +\infty. \tag{2.10}$$

As

$$\lim_{h \to \infty} T_h = a, \qquad \lim_{h \to 0} T_h = \infty,$$

and T_h is a decreasing function of h, there exists a unique $h_c > 0$ such that

$$L = T_{h_c} - a = \sqrt{\frac{p+1}{2}}\,h_c^{-\frac{p-1}{2}} \int_1^\infty \frac{d\theta}{\sqrt{\theta^{p+1} - 1}}. \tag{2.11}$$

Obviously, $T_h > a + L$ if $h < h_c$, whereas $T_h < a + L$ if $h > h_c$. Therefore, the solution of (2.7) is defined in $[a, a+L]$ if, and only if $h < h_c$, while it blows up at $T_h < a + L$ if $h > h_c$. When $h = h_c$, the solution blows up exactly at $x = a + L$. Consequently, any solution of (2.2) must satisfy $u(a) < h_c$. Subsequently, we denote by u_h the unique solution of (2.7). We claim that

$$u_{h_1}(x) < u_{h_2}(x) \quad \text{for all } \ x \in [a, a+L] \quad \text{if } \ 0 < h_1 < h_2 < h_c. \tag{2.12}$$

If not, there exists $x_0 \in (a, a+L)$ such that

$$u'_{h_1}(x_0) \geq u'_{h_2}(x_0), \quad u_{h_1}(x_0) = u_{h_2}(x_0), \quad u_{h_1}(x) < u_{h_2}(x)$$

for all $x \in (a, x_0)$. Then, from (2.8) we find that

$$u'_{h_1}(x_0) = \int_a^{x_0} u_{h_1}^p(y)dy < \int_a^{x_0} u_{h_2}^p(y)dy = u'_{h_2}(x_0),$$

a contradiction. Moreover, by continuous dependence,

$$\lim_{h\uparrow h_c} u_h(x) = u_{h_c}(x) \quad \text{for all } \ x \in [a, a+L), \tag{2.13}$$

with uniform convergence in compact subintervals of $[a, a+L)$, by the Ascoli-Arzela theorem. Naturally,

$$\lim_{h\uparrow h_c} u_h'(a+L) = \infty. \tag{2.14}$$

Indeed, due to (2.8), (2.12) and (2.13), we find from the Lebesgue monotone convergence theorem that

$$\lim_{h\uparrow h_c} u_h'(a+L) = \int_a^{a+L} u_{h_c}^p(y)\, dy. \tag{2.15}$$

On the other hand, by [14, Th. 1.1], there is a positive constant $C > 0$ such that

$$\lim_{x\uparrow a+L} \frac{u_{h_c}(x)}{C(a+L-x)^{-2/(p-1)}} = 1.$$

The fact that

$$\int_a^{a+L} (a+L-y)^{-2p/(p-1)}\, dy = +\infty$$

yields (2.14).

2.3. The profile of the symmetric solutions in $(a+L, m)$**.** The next result provide us with the behavior of the symmetric solutions of (1.5) in $(a+L, m)$ for the choice (2.1).

PROPOSITION 2.4. *For every $h \in (0, h_c)$ there exists a unique $\kappa = \kappa(h) > u_h^{p-1}(a+L)$ such that the unique solution of the problem*

$$\begin{cases} -w'' = \kappa w - w^p & \textit{in } \ (a+L, m), \\ w(a+L) = u_h(a+L), & w'(m) = 0, \end{cases} \tag{2.16}$$

also satisfies $w'(a+L) = u_h'(a+L)$. Consequently, (2.4) *possesses a solution, which is unique, by Lemma 2.2. Moreover, $\kappa(h)$ is a continuous function of $h \in (0, h_c)$.*

Proof. For every $h \in (0, h_c)$ and $\kappa \in \mathbb{R}$, (2.16) admits a positive solution, because $\underline{w} := 0$ is a subsolution and $\overline{w} := M$ is a supersolution for sufficiently large $M > 0$. Moreover, by adapting the proof of Lemma 2.2, the solution must be unique.

Suppose $\kappa = u_h^{p-1}(a+L)$. Then, the constant $w := u_h(a+L)$ itself provides us with the unique solution of (2.16). As $w'(a+L) = 0 < u_h'(a+L)$, to complete the proof of the existence it suffices to show that

$$\lim_{\kappa\uparrow\infty} w'(a+L) = \infty. \tag{2.17}$$

Indeed, as $w'(a+L)$ varies continuously with κ, (2.17) guarantees the existence of $\kappa = \kappa(h) > u_h^{p-1}(a+L)$ such that $w'(a+L) = u_h'(a+L)$, which concludes the proof. To prove (2.17) we perform the change of variable $w = \kappa^{1/(p-1)}\xi$, which transforms (2.16) in the problem

$$\begin{cases} -\frac{1}{\kappa}\xi'' = \xi - \xi^p & \text{in } \ (a+L, m), \\ \xi(a+L) = \kappa^{-1/(p-1)} u_h(a+L), & \xi'(m) = 0. \end{cases} \tag{2.18}$$

For sufficiently large κ, the constant function $\overline{\xi} := 1$ provides us with a strict supersolution of (2.18), and $\underline{\xi} := v_\kappa|_{[a+L,m]}$, where v_κ stands for the unique positive solution of

$$\begin{cases} -\frac{1}{\kappa}v'' = v - v^p & \text{in } (a+L, b-L), \\ v(a+L) = v(b-L) = 0, \end{cases} \tag{2.19}$$

provides us with a strict subsolution of (2.18). Thus, by comparison and uniqueness, we find that $v_\kappa < \xi < 1$ in $[a+L, m]$. On the other hand, by [8, Th. 2.1], we already know that

$$\lim_{\kappa \uparrow \infty} v_\kappa = 1 \quad \text{uniformly in } (a+L+\epsilon, b-L-\epsilon)$$

for all sufficiently small $\epsilon > 0$. Consequently,

$$\lim_{\kappa \uparrow \infty} \xi = 1 \quad \text{uniformly in } (a+L+\epsilon, m]$$

for all $\epsilon > 0$, $\epsilon \sim 0$. As $\xi(a+L) \to 0$ for $\kappa \to \infty$, it becomes apparent that ξ develops a boundary layer at $a+L$. Moreover, as $\xi < 1$ implies $\xi'' < 0$, ξ' must be decreasing. Therefore, $\lim_{k\to\infty} \xi'(a+L) = \infty$, which implies (2.17) and ends the proof of the existence.

The uniqueness of $\kappa(h)$ is an easy consequence from the fact that the mapping $\kappa \mapsto w'(a+L)$ is increasing for $\kappa > u_h^{p-1}(a+L)$. Indeed, according to the theorem of differentiation of Peano, this map is real analytic and the function $\psi := \partial_\kappa w$ satisfies

$$\begin{cases} \left(-D^2 + pw^{p-1} - \kappa\right)\psi = w & \text{in } (a+L, m), \\ \psi(a+L) = \psi'(m) = 0. \end{cases} \tag{2.20}$$

According to (2.16), we have that

$$\left(-D^2 + pw^{p-1} - \kappa\right) w = (p-1)w^p > 0 \quad \text{in } (a+L, m).$$

Moreover, $w(a+L) > 0$ and $w'(m) = 0$. Thus, $w > 0$ provides us with a positive strict supersolution of the differential operator $-D^2 + pw^{p-1} - \kappa$ in the interval $(a+L, m)$ subject to the boundary conditions $\mathfrak{B}w = 0$, where $\mathfrak{B}$ is defined by

$$\mathfrak{B}u(a+L) = u(a+L), \qquad \mathfrak{B}u(m) = u'(m).$$

Consequently, thanks to, e.g., [2, Th. 2.4], or [15, Th. 7.10], we find that $-D^2 + pw^{p-1} - \kappa$ satisfies the strong maximum principle in $(a+L, m)$ subject to the boundary operator $\mathfrak{B}$. Hence, (2.20) implies $\psi \gg 0$ and, consequently, $\partial_k w'(a+L) > 0$. This shows the uniqueness of $\kappa(h)$. The continuity of $\kappa(h)$ is a byproduct of the uniqueness and the continuous dependence of the solutions of (2.16) with respect to κ and h. The proof is complete. □

2.4. The proof of Theorem 2.1. According to Proposition 2.4, (2.2) admits a positive solution if, and only if, for some $h \in (0, h_c)$, $2\lambda \int_a^{a+L} u_h = \kappa(h)$. Therefore, the set of λ's for which (2.2), and hence (1.5) under assumptions (2.1), admits a positive solution is given by the image of the map

$$\Lambda(h) := \frac{\kappa(h)}{2\int_a^{a+L} u_h} \qquad 0 < h < h_c. \tag{2.21}$$

According to Proposition 2.4, by the continuous dependence of u_h with respect to h, it becomes apparent that $\Lambda(h)$ is a continuous function of h. Moreover, by construction,

$$\kappa(h) > u_h^{p-1}(a+L) \tag{2.22}$$

and, owing to (2.8) and (2.9), we have that

$$u_h'(a+L) = \int_a^{a+L} u_h^p(y)\,dy = \sqrt{\frac{2}{p+1}\left(u_h^{p+1}(a+L) - h^{p+1}\right)}. \tag{2.23}$$

Thus, by Hölder inequality, we find from (2.23) that

$$\begin{aligned}\int_a^{a+L} u_h(y)\,dy &\le L^{1/q}\left(\int_a^{a+L} u_h^p(y)\,dy\right)^{1/p} = L^{1/q}\left(u_h'(a+L)\right)^{1/p}\\ &= L^{1/q}\left[\frac{2}{p+1}\left(u_h^{p+1}(a+L) - h^{p+1}\right)\right]^{\frac{1}{2p}}\end{aligned}$$

and consequently, for every $h \in (0, h_c)$, we have that

$$\Lambda(h) \ge \frac{u_h^{p-1}(a+L)}{2L^{1/q}\left[\frac{2}{p+1}\left(u_h^{p+1}(a+L) - h^{p+1}\right)\right]^{\frac{1}{2p}}}. \tag{2.24}$$

As $p-1 > \frac{p+1}{2p}$ if $p \ge 2$ and $\lim_{h\uparrow h_c} u_h(a+L) = +\infty$, we find from (2.24) that $\lim_{h\uparrow h_c}\Lambda(h) = +\infty$. Since $\lim_{h\to 0}\Lambda(h) = \lambda_c(p) \ge 0$ and Λ is continuous, $\Lambda((0,h_c)) = (\lambda_c(p), +\infty)$. Hence, there exists a $h \in (0, h_c)$ such that $\Lambda(h) = \lambda$. This ends the proof.

3. Some one-dimensional numerical experiments for Ω_0 large. In this section we make the special choice

$$\Omega = (-1, 1), \qquad \Omega_0 = (-0.6, 0.6),$$

in (2.1) and couple a pseudo-spectral method with a path-following solver to compute the bifurcation diagram of the positive solutions of (1.5) for $p = 2$ and $p = 3$. This method is extremely efficient at a very low computational cost. The reader is sent to [16] and [17] for the technical details concerning the numerics.

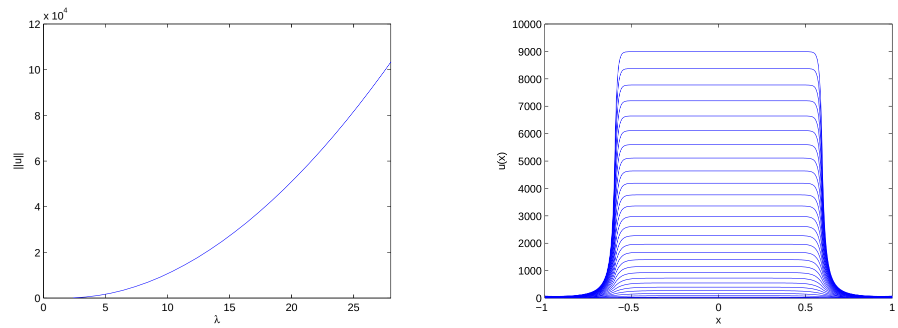

FIG. 3.1. *Global bifurcation diagram for $p = 2$ and profiles of a series of solutions along it.*

Figure 3.1 shows a plot of the global bifurcation diagram for $p = 2$ on the left picture together with a superimposed series of plots of positive solutions of (1.5) along the solution curve on the left for an increasing series of values of λ on the right picture. The curve of positive solutions bifurcates from $u = 0$ at the critical value of the parameter $\lambda \sim 2.083214$ and it is defined for all further value of the parameter λ for which we have tried to compute it.

Thus, using the notations of Theorem 2.1, it seems that $\lambda_c(2) \sim 2.083214$. In the bifurcation diagram we are plotting λ versus the ℓ^2-norm of the computed solution u, which has been represented by $\|u\|$ in ordinates. All computed positive solutions along the bifurcated curve are linearly stable as steady-states of the associated parabolic problem, and increase with λ.

Although the apparent global concavity exhibited by the first picture of Figure 3.1 suggests that the curve of positive solutions emanates tangentially to the trivial state, with a *bifurcation direction* very similar to the one of the λ-axis, the magnified diagrams around the bifurcation point from the trivial state shown in Figures 3.2 and 3.4 show that actually the curve loses its concavity after some value of λ, close to λ_c, and it becomes convex for all further values of the parameter for which we computed it. Actually, as illustrated by the first plot of Figure 3.4, the bifurcation of the curve from zero is almost vertical.

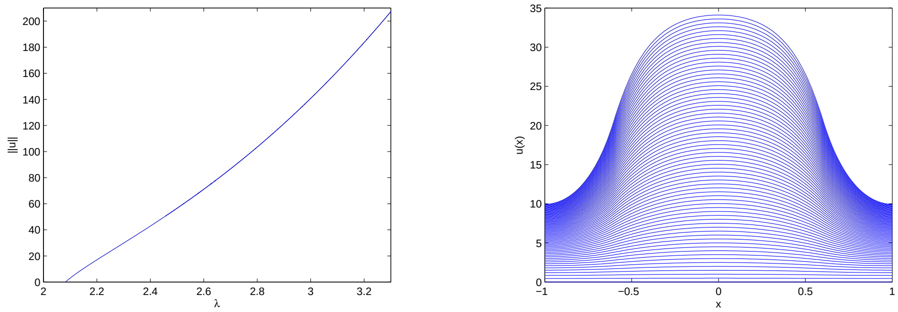

FIG. 3.2. *Magnified diagram for $p = 2$ and profiles of a series of solutions along it.*

Our numerical experimentes reveal that these general patterns change when $p > 2$. Actually, according to the numerics, the behavior of the small positive solutions of the model change drastically when $p > 2$, as the curve always bifurcates from $u = 0$ at $\lambda = 0$. Consequently, we conjecture that $\lambda_c(p) = 0$ for all $p > 2$ in Theorem 2.1. Figure 3.2 shows the computed bifurcation diagram for $p = 3$. It bifurcates from $u = 0$ with a very high slope, as illustrated by the second picture of Figure 3.4, at the critical value of the parameter $\lambda = 0$.

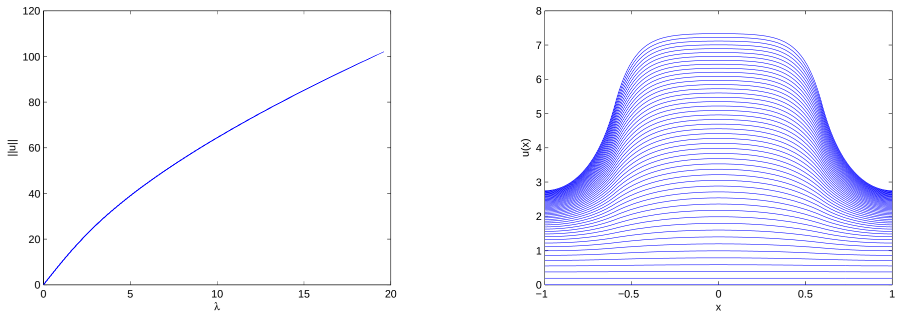

FIG. 3.3. *Global bifurcation diagram for $p = 3$ and a series of solutions along it.*

As in the problem (1.5) the parameter λ lies in front of the first term of the nonlinearity, the bifurcation of positive solutions from $u = 0$ cannot be expected to be *based on the linear part* of the equation, as it occurs in most of the available literature on bifurcation problems, where λ is a sort of spectral parameter, but, instead, on their nonlinear terms. Actually, $u = 0$

is a linearly neutrally stable equilibrium of (1.1) for all $\lambda \geq 0$, as zero is always an eigenvalue of the linearized equation at $u = 0$. Consequently, the fact that bifurcation occurs as λ crosses 2.083214 in case $p = 2$, or $\lambda = 0$ in case $p > 2$, is based on the nonlinear terms of (1.5).

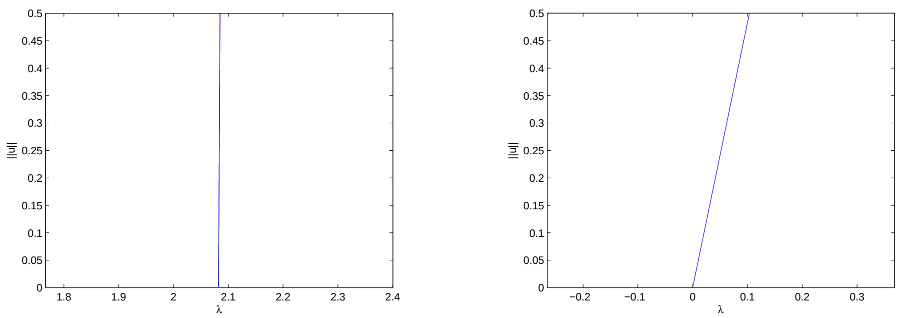

FIG. 3.4. *Magnifying the curves around the bifurcation points in cases* $p = 2$ *(left plot) and* $p = 3$ *(right plot).*

Although all the positive solutions of (1.5) are point-wise increasing with λ, they grow faster in Ω_0 than in $\Omega \setminus \Omega_0$, where, according to the maximum principle, they are bounded above by the unique solution of (2.7) for the special choice $h = h_c$ (see (2.11) for the definition of h_c, if necessary). Actually, as illustrated by Figure 3.5, the solutions stabilize in $\Omega \setminus \bar{\Omega}_0$ to the unique positive large solution of $u'' = u^2$ in $(-1, -0.6)$ with $u'(-1) = 0$.

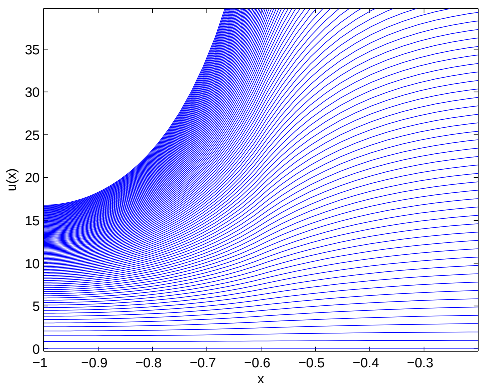

FIG. 3.5. *Stabilization of the solutions in* $(-1, -0.6)$ *as* λ *grows.*

Based on the numerics, we conjecture that for any sequence of positive solutions of (1.5), say (λ_n, u_n), $n \geq 1$, with $\lim_{n\to\infty} \lambda_n = \infty$, the following holds

$$\lim_{n\to\infty} u_n = \begin{cases} \infty & \text{in } \bar{\Omega}_0, \\ \ell & \text{in } \Omega \setminus \bar{\Omega}_0, \end{cases}$$

where ℓ stands for the minimal positive solution of the singular boundary value problem

$$\begin{cases} \Delta u = u^p & \text{in } \Omega \setminus \bar{\Omega}_0, \\ \partial_n u|_{\partial\Omega} = 0, \quad u|_{\partial\Omega_0} = +\infty, \end{cases} \tag{3.1}$$

whose existence is guaranteed by [14, Prop. 3.4].

4. Admissible bifurcation values from $u = 0$. The next result explains some of the most significant differences observed in Section 3 between the cases $p = 2$ and $p > 2$.

THEOREM 4.1. *Suppose $p \geq 2$ and $\lambda_0 \in \mathbb{R}$ is a bifurcation value to positive solutions of* (1.5) *from $(\lambda, u) = (\lambda, 0)$, i.e., there exists a sequence of positive solutions of* (1.5)*, say (λ_n, u_n), $n \geq 1$, such that $\lambda_n \to \lambda_0$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Then, $\lambda_0 = 0$ if $p > 2$, while $\lambda_0 = \lambda^*$ if $p = 2$, where*

$$\lambda^* := \frac{|\Omega|}{|\Omega_0||\Omega \setminus \Omega_0|}. \tag{4.1}$$

When $N = 1$, $\Omega = (-1, 1)$ and $\Omega_0 = (-0.6, 0.6)$, we have that

$$\lambda^* = \frac{|\Omega|}{|\Omega_0||\Omega \setminus \Omega_0|} = \frac{2}{1.2 \times 0.8} = 2.083333\ldots$$

indeed approximates the value of $\lambda_c(2) \sim 2.083214$ already computed in Section 3.

Proof. Integrating in Ω the differential equation yields

$$\lambda_n \int_{\Omega_0} u_n \int_{\Omega\setminus\Omega_0} u_n = \int_\Omega u_n^p, \qquad n \geq 1. \tag{4.2}$$

Moreover, dividing by $\|u_n\|_\infty$ and setting

$$v_n := u_n/\|u_n\|_\infty, \qquad n \geq 1,$$

we find that

$$-\Delta v_n = \left(\lambda_n V(x) \int_{\Omega\setminus\Omega_0} u_n - u_n^{p-1}\right) v_n \tag{4.3}$$

for all $n \geq 1$. Since

$$\|v_n\|_\infty = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \|\lambda_n V(x) \int_{\Omega\setminus\Omega_0} u_n - u_n^{p-1}\|_\infty = 0,$$

by elliptic regularity and the inherent associated compactness, one can extract a subsequence, also labelled by n, such that $\lim_{n\to\infty} v_n = \psi$ in $L^\infty(\Omega)$ for some smooth ψ. As $\psi > 0$, $\|\psi\|_\infty = 1$, and letting $n \to \infty$ in (4.3) we obtain that $-\Delta\psi = 0$, necessarily $\psi = 1$ and therefore, $\lim_{n\to\infty} v_n = 1$ in $L^\infty(\Omega)$. Consequently, it becomes apparent from (4.2) that

$$\lambda_0 = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{\int_\Omega u_n^{p-2} v_n^2}{\int_{\Omega_0} v_n \int_{\Omega\setminus\Omega_0} v_n} = 0$$

if $p > 2$, while $\lambda_0 = \lambda^*$ if $p = 2$. □

5. The multidimensional problem with $p > 2$. In this section we will study the general multidimensional problem (1.5) with $N \geq 1$ and $p > 2$. Our main result is the following.

THEOREM 5.1. *Suppose $p > 2$. Then,* (1.5) *has a positive solution if and only if $\lambda > 0$.*

Proof. Suppose (1.5) possesses a positive solution. Then, we already know that $\lambda > 0$. Subsequently, we fix $\lambda > 0$ and consider the auxiliary problem

$$\begin{cases} -\Delta u = \mu u + \lambda V(x) \left(\int_{\Omega\setminus\Omega_0} u(y)\, dy\right) u - u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\mu \in \mathbb{R}$ is regarded as an auxiliary (primary) bifurcation parameter from the *trivial* solution $(\mu, u) = (\mu, 0)$.

Let E denote the closed subspace of the Banach space $\mathcal{C}^1(\bar{\Omega})$ formed by all functions $u \in \mathcal{C}^1(\bar{\Omega})$ such that $\partial_n u = 0$ on $\partial\Omega$. Naturally, the solutions of (5.1) are the fixed points of the compact operator $\mathfrak{K} : \mathbb{R} \times E \to E$ defined by

$$\mathfrak{K}(\mu, u) := (-\Delta + 1)^{-1} \left[(\mu + 1)u + \lambda V(x) \left(\int_{\Omega \setminus \Omega_0} u(y)dy \right) u - u^p \right] \tag{5.2}$$

for all $\mu \in \mathbb{R}$ and $u \in E$, where $(-\Delta + 1)^{-1}$ is the resolvent of $-\Delta + 1$ under homogeneous Neumann boundary conditions on $\partial\Omega$. Consequently, the solutions of (5.1) are the zeroes of

$$\mathfrak{F}(\mu, u) := u - \mathfrak{K}(\mu, u), \qquad (\mu, u) \in \mathbb{R} \times E. \tag{5.3}$$

The operator $\mathfrak{F}$ is of class $\mathcal{C}^2$ and, since it is a compact perturbation of the identity of E, it is Fredholm of index zero. Moreover, it can be expressed in the form

$$\mathfrak{F}(\mu, u) = \mathfrak{L}(\mu)u + \mathfrak{N}(u), \qquad (\mu, u) \in \mathbb{R} \times E,$$

where

$$\mathfrak{L}(\mu)u = D_u\mathfrak{F}(\mu, 0)u = u - (\mu + 1)(-\Delta + 1)^{-1}u, \qquad (\mu, u) \in \mathbb{R} \times E,$$

and

$$\mathfrak{N}(u) = (-\Delta + 1)^{-1} \left(u^p - \lambda V(x) \left(\int_{\Omega \setminus \Omega_0} u(y)dy \right) u \right)$$

for all $u \in E$. Note that

$$\mathfrak{F}(\mu, 0) = 0, \qquad \mathfrak{N}(0) = 0, \qquad D_u\mathfrak{N}(0) = 0,$$

for all $\mu \in \mathbb{R}$. Moreover, setting

$$\mathfrak{L}_0 := \mathfrak{L}(0), \qquad \mathfrak{L}_1 := \frac{d\mathfrak{L}}{d\mu}(0) = -(-\Delta + 1)^{-1},$$

we have that

$$\ker \mathfrak{L}_0 = \operatorname{span}[1], \qquad \mathfrak{L}_1 1 \notin \operatorname{Im} \mathfrak{L}_0. \tag{5.4}$$

The first identity of (5.4) is obvious. Suppose the second one does not hold. Then, there exists $u \in E$ such that

$$\mathfrak{L}_0 u = u - (-\Delta + 1)^{-1}u = \mathfrak{L}_1 1 = -(-\Delta + 1)^{-1}1.$$

By elliptic regularity, u must be a strong solution of $\Delta u = 1$ and, integrating in Ω, yields $0 = \int_\Omega 1 = |\Omega|$, which is impossible. Therefore, (5.4) is satisfied and hence, the transversality condition of Crandall and Rabinowitz [3] holds.

Let Y denote the closed subspace of E defined by

$$Y := \left\{ u \in E \ : \ \int_\Omega u = 0 \right\}.$$

Due to (5.4),

$$\ker \mathfrak{L}_0 \oplus Y = E$$

and consequently, by the main theorem of [3], there exist $\epsilon > 0$, $\rho > 0$, and two (unique) maps of class $\mathcal{C}^1$, $\mu : (-\epsilon, \epsilon) \to \mathbb{R}$ and $y : (-\epsilon, \epsilon) \to Y$, such that

$$\mu(0) = 0, \quad y(0) = 0, \quad \mathfrak{F}(\mu(s), s(1 + y(s))) = 0 \qquad \forall\, s \in (-\epsilon, \epsilon),$$

and

$$\mathfrak{F}^{-1}(0) \cap B_\rho = \{(\mu, 0) \ : \ \mu \sim 0\} \cup \{(\mu(s), s(1 + y(s))) \ : \ s \sim 0\},$$

where B_ρ is the ball of $\mathbb{R} \times E$ with radius ρ centered at zero. By elliptic regularity, $(\mu(s), s(1+y(s)))$ is a strong solution of (5.1) for all $s \in (-\epsilon, \epsilon)$. Thus, dividing by s and setting

$$y_1 := y'(0), \qquad \mu_1 := \mu'(0),$$

we find that

$$\begin{aligned} -\Delta(1 + y_1 s + o(s)) = [\mu_1 s + o(s) + \lambda V(x) \int_{\Omega\setminus\Omega_0} & s(1 + s y_1 + o(s)) \\ & - s^{p-1}(1 + y(s))^{p-1}](1 + s y_1 + o(s)) \end{aligned}$$

for all $s \in (-\epsilon, \epsilon)$. Hence, identifying the coefficients of s, yields

$$-\Delta y_1 = \mu_1 + \lambda |\Omega \setminus \Omega_0| V(x)$$

and, therefore, integrating in Ω, we obtain that

$$\mu_1 = -\lambda \frac{|\Omega \setminus \Omega_0||\Omega_0|}{|\Omega|} < 0. \tag{5.5}$$

Consequently, the bifurcation to positive solutions is subcritical in μ for all $\lambda > 0$.

The second relation of (5.4) can be equivalently expressed as

$$\mathfrak{L}_1 (\ker \mathfrak{L}_0) \oplus \operatorname{Im} \mathfrak{L}_0 = E$$

and hence, the concept of algebraic multiplicity $\chi[\mathfrak{L}; \cdot]$ introduced by Esquinas and López-Gómez [7] and later refined in [13] and in López-Gómez and Mora-Corral [18], satisfies $\chi[\mathfrak{L}(\mu); 0] = 1$. Consequently, thanks to Theorems 6.0.1, 8.1.1 and Proposition 12.3.1 of [18], the local topological index $\operatorname{Ind}(\mathfrak{L}(\mu), 0)$ changes as μ crosses 0. Therefore, by Theorems 6.2.1 and 6.3.1 of [13], there is a component $\mathfrak{C}_0$ of the set of non-trivial solutions of (5.1) with $(\mu, u) = (0, 0) \in \bar{\mathfrak{C}}_0$ satisfying the global alternative of Rabinowitz [21]. According to the main theorem of Crandall and Rabinowitz [3], $\mathfrak{C}_0$ must be a curve of class $\mathcal{C}^1$ in a neighborhood of $(\mu, u) = (0, 0)$. It should be noted that the change of the local index of $\mathfrak{L}(\mu)$ cannot be inferred directly from [3], but from [7], [13] and [18].

According to the unilateral theory of Rabinowitz [21], in a neighborhood of $(0, 0)$, the component $\mathfrak{C}_0$ consists of two subcomponents: $\mathfrak{C}_0^+$, formed by positive solutions, and $\mathfrak{C}_0^-$, filled in by negative ones. As explained in [13, Chapter 6], Dancer [5], and López-Gómez and Molina-Meyer [16], $\mathfrak{C}_0^+$ does not necessarily satisfies the *global alternative* of Rabinowitz [21], as it does the whole component $\mathfrak{C}_0$. However, due to [16, Th. 1.1], there exists an unbounded subcomponent $\mathfrak{C}_+$ of the set of positive solutions of (5.1) such that $(\mu, 0) \in \bar{\mathfrak{C}}_+$ if, and only if, $\mu = 0$.

Subsequently, we will show that there exist $\mu_0 < 0$ and $C > 0$ such that (5.1) cannot admit a positive solution if $\mu \leq \mu_0$ and

$$\|u\|_\infty \leq C \tag{5.6}$$

for any positive solution (μ, u) of (5.1) with $\mu \leq 0$. This shows that $\mathfrak{C}_+$ must be bounded in $[\mu_0, 0] \times E$ and, therefore, (1.5) must have a positive solution.

Suppose (μ, u) is a positive solution of (5.1) with $\mu \leq 0$. Then, for every $x \in \Omega \setminus \Omega_0$, we find from (5.1) that

$$-\Delta u(x) = \mu u(x) - u^p(x) \leq -u^p(x) < 0.$$

Thus, setting $v := -u < 0$, we have that $-\Delta v > 0$ in $\Omega \setminus \Omega_0$. Hence, according to the minimum principle,

$$\inf_{\Omega \setminus \Omega_0} v = \min_{\partial\Omega \cup \partial\Omega_0} v < 0$$

cannot be reached in $\Omega \setminus \Omega_0$. Indeed, in the contrary case, v, and hence u, should be constant and so, $u^{p-1} = \mu \leq 0$, which is imposible. Moreover, by the Hopf boundary lemma, we must have $\frac{\partial v}{\partial n}(x) < 0$ for all $x \in \partial\Omega \cup \partial\Omega_0$ such that

$$u(x) = \sup_{\Omega \setminus \Omega_0} u \tag{5.7}$$

(see, e.g., Protter and Weinberger [20], or López-Gómez [15, Chapter 1]). As $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, (5.7) implies that $x \in \partial\Omega_0$ and $\frac{\partial u}{\partial n}(x) > 0$. Consequently, there exists $x_0 \in \Omega_0$ such that

$$u(x_0) = \|u\|_\infty.$$

As $u \in \mathcal{C}^2(\Omega_0)$, we find that $\Delta u(x_0) \leq 0$ and hence,

$$u^{p-1}(x_0) = \|u\|_\infty^{p-1} \leq \mu + \lambda \int_{\Omega \setminus \Omega_0} u \leq \lambda u(x_0) |\Omega \setminus \Omega_0|, \tag{5.8}$$

which implies

$$\|u\|_\infty \leq (\lambda |\Omega \setminus \Omega_0|)^{\frac{1}{p-2}} \tag{5.9}$$

and provides us with the estimate (5.6). Finally, we will show the existence of μ_0. Integrating the differential equation of (5.1) in Ω yields

$$\mu \int_\Omega u = -\lambda \int_{\Omega_0} u \int_{\Omega \setminus \Omega_0} u + \int_\Omega u^p > -\lambda \int_{\Omega_0} u \int_{\Omega \setminus \Omega_0} u > -\lambda \left(\int_\Omega u \right) \|u\|_\infty |\Omega \setminus \Omega_0|$$

because $\lambda > 0$ and $u \gg 0$ in Ω. Thus, dividing by $\int_\Omega u$ and using (5.9), we find that

$$\mu > -\lambda \|u\|_\infty |\Omega \setminus \Omega_0| \geq -(\lambda |\Omega \setminus \Omega_0|)^{1 + \frac{1}{p-2}}.$$

Consequently, we can take $\mu_0 = -(\lambda |\Omega \setminus \Omega_0|)^{\frac{p-1}{p-2}}$. This ends the proof. □

It the final part of the previous proof, it should be noted that the estimate

$$\mu > -\lambda \|u\|_\infty |\Omega \setminus \Omega_0| \tag{5.10}$$

is also valid for $p = 2$.

6. The multidimensional case $p = 2$. Adapting the proof of Theorem 5.1 yields

THEOREM 6.1. *Suppose $p = 2$ and (1.5) admits a positive solution. Then,*

$$\lambda \geq \frac{2}{\sqrt{|\Omega_0||\Omega \setminus \Omega_0|}}. \tag{6.1}$$

Moreover, for every $\lambda > 0$, there exists a component $\mathfrak{C}_+$ of the set of positive solutions of (5.1) emanating from $u = 0$ at $\mu = 0$. Actually, in a neighborhood of $(\mu, u) = (0, 0)$, the component $\mathfrak{C}_+$ consists of a differentiable curve $(\mu(s), s(1 + y(s)))$, $s > 0$, $s \sim 0$, such that

$$\mu(s) = \left(1 - \lambda\frac{|\Omega_0||\Omega \setminus \Omega_0|}{|\Omega|}\right) s + o(s) \qquad s \downarrow 0. \tag{6.2}$$

In addition, if $\lambda = \lambda^*$, where λ^* is given by (4.1), then $\mu'(0) = 0$ and

$$\mu''(0) = 2\left(\frac{1}{|\Omega_0|} - \frac{1}{|\Omega \setminus \Omega_0|}\right)\int_{\Omega\setminus\Omega_0} y_1, \tag{6.3}$$

where $'$:= d/ds and y_1 stands for the unique solution of

$$-\Delta y_1 = \frac{|\Omega|}{|\Omega_0|}V(x) - 1, \qquad \int_\Omega y_1 = 0, \qquad \left.\frac{\partial y_1}{\partial n}\right|_{\partial\Omega} = 0. \tag{6.4}$$

Furthermore, this function satisfies

$$\int_{\Omega\setminus\Omega_0} y_1 < 0. \tag{6.5}$$

Proof. Suppose (1.5) possesses a positive solution u. Then, (1.6) holds with $p = 2$. Thus, by Hölder inequality,

$$\int_\Omega u^2 = \lambda\int_{\Omega_0} u\int_{\Omega\setminus\Omega_0} u \leq \lambda\sqrt{|\Omega_0||\Omega \setminus \Omega_0|}\left(\int_{\Omega_0} u^2\right)^{\frac{1}{2}}\left(\int_{\Omega\setminus\Omega_0} u^2\right)^{\frac{1}{2}}$$

and setting $x := \left(\int_{\Omega_0} u^2\right)^{1/2} > 0$ and $y := \left(\int_{\Omega\setminus\Omega_0} u^2\right)^{1/2}$ we can express the previous inequality in the form

$$x^2 + y^2 \leq \lambda xy\sqrt{|\Omega_0||\Omega \setminus \Omega_0|} \iff \frac{x}{y} + \frac{y}{x} \leq \lambda\sqrt{|\Omega_0||\Omega \setminus \Omega_0|}$$

and therefore,

$$2 = \inf_{t>0}\left(t + t^{-1}\right) \leq \lambda\sqrt{|\Omega_0||\Omega \setminus \Omega_0|},$$

which concludes the proof that (6.1) is necessary for the existence.

The existence of $\mathfrak{C}_+$, as well as its local structure near $(\mu, u) = (0, 0)$, follows by adapting the proof of Theorem 5.1. Actually, using the same notations as therein, we find that

$$-\Delta(1 + y(s)) = [\mu(s) + \lambda V(x)\int_{\Omega\setminus\Omega_0} s(1 + y(s)) - s(1 + y(s))](1 + y(s))$$

for all $s \in (-\epsilon, \epsilon)$. Thus, setting

$$y(s) = sy_1 + s^2y_2 + o(s^2), \qquad \mu(s) = s\mu_1 + s^2\mu_2 + o(s^2), \qquad \text{as} \quad s \to 0,$$

and identifying the terms of order one in s, we find that

$$-\Delta y_1 = \mu_1 - 1 + \lambda|\Omega \setminus \Omega_0|V(x).$$

Therefore, integrating in Ω yields

$$\mu_1 = 1 - \lambda\frac{|\Omega \setminus \Omega_0||\Omega_0|}{|\Omega|} = 1 - \frac{\lambda}{\lambda^*} \tag{6.6}$$

which ends the proof of (6.2). It should be noted that, since Y is a closed subspace of E, we have that $y_1, y_2 \in Y$ and, consequently, $\frac{\partial y_j}{\partial n} = 0$ on $\partial\Omega$ and $\int_\Omega y_j = 0$ for each $j \in \{1,2\}$. In particular, $\mu_1 = 0$ if $\lambda = \lambda^*$, and y_1 is the unique solution of (6.4). To conclude the proof, it remains to find out the value of μ_2. Identifying terms of order two in s, we find that

$$-\Delta y_2 = \left(\frac{|\Omega|}{|\Omega_0|}V(x) - 1\right) y_1 + \mu_2 + \lambda^* V(x) \int_{\Omega\setminus\Omega_0} y_1 - y_1$$

and, therefore, integrating in Ω yields

$$\frac{|\Omega|}{|\Omega_0|}\int_{\Omega_0} y_1 - \int_\Omega y_1 + \mu_2|\Omega| + \frac{|\Omega|}{|\Omega \setminus \Omega_0|}\int_{\Omega\setminus\Omega_0} y_1 - \int_\Omega y_1 = 0.$$

Consequently, since $0 = \int_\Omega y_1 = \int_{\Omega_0} y_1 + \int_{\Omega\setminus\Omega_0} y_1 = 0$, we find that

$$\mu_2 = \left(\frac{1}{|\Omega_0|} - \frac{1}{|\Omega \setminus \Omega_0|}\right)\int_{\Omega\setminus\Omega_0} y_1, \tag{6.7}$$

which shows (6.3). Finally, multiplying the differential equation of (6.4) by y_1 and integrating by parts in Ω yields

$$0 < \int_\Omega |\nabla y_1|^2 = \frac{|\Omega|}{|\Omega_0|}\int_{\Omega_0} y_1 - \int_\Omega y_1 = \frac{|\Omega|}{|\Omega_0|}\int_{\Omega_0} y_1.$$

Therefore, $\int_{\Omega\setminus\Omega_0} y_1 = -\int_{\Omega_0} y_1 < 0$, which shows (6.5) and ends the proof. □

According to (6.6), we have that $\mu_1 > 0$ if, and only if, $\lambda < \lambda^*$ and that $\mu_1 < 0$ if, and only if, $\lambda > \lambda^*$. Thus, $\mathfrak{C}_+$ bifurcates from the trivial solution $u = 0$ sub-critically if $\lambda > \lambda^*$, while it bifurcates super-critically if $\lambda < \lambda^*$. This supports the validity of the next result.

THEOREM 6.2. *Suppose* $p = 2$. *Then,* (1.5) *has a positive solution for each* $\lambda > \lambda^*$.

Proof. As the proof of Theorem 5.1, the proof of this result is also based on the existence of a priori bounds for the positive solutions of (5.1) with $\mu \le 0$, and in the fact $\mu_1 < 0$ and $\mu > \mu_0$, for some constant $\mu_0 < 0$, if $\lambda > \lambda^*$ and (5.1) admits a positive solution. On the contrary, suppose that there is a sequence of positive solutions, say (μ_n, u_n), $n \ge 1$, with $\mu_n \le 0$, such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, setting

$$\tilde u_n := u_n/\|u\|_\infty, \qquad n \ge 1,$$

it follows from (5.1) that

$$\begin{cases} -\frac{1}{\|u_n\|_\infty}\Delta \tilde u_n = \frac{\mu_n}{\|u_n\|_\infty}\tilde u_n + \lambda V(x)\left(\int_{\Omega\setminus\Omega_0}\tilde u_n\right)\tilde u_n - \tilde u_n^2 \qquad \text{in } \Omega, \\ \frac{\partial \tilde u_n}{\partial n}\big|_{\partial\Omega} = 0. \end{cases} \tag{6.8}$$

Let K be an arbitrary compact subset of $\Omega \setminus \bar{\Omega}_0$, and consider $x_{n,K} \in K$ such that

$$u_n(x_{n,K}) = \max_K u_n, \qquad n \geq 1.$$

As $K \subset \Omega \setminus \bar{\Omega}_0$, there exists a constant $r > 0$ such that $B_r(x_{n,K}) \subset \Omega \setminus \bar{\Omega}_0$ for all $n \geq 1$. Subsequently, we set

$$\epsilon_n := \|u_n\|_\infty^{-1/2}, \quad y := (x - x_{n,K})/\epsilon_n, \qquad v_n(y) := u_n(x)/\|u_n\|_\infty, \quad n \geq 1. \tag{6.9}$$

As K is compact, we can assume, without loss of generality, that there exists $x_{\infty,K} \in K$ such that $x_{n,K} \to x_{\infty,K}$ as $n \to \infty$. According to (1.2), (6.8) and (6.9), we find that

$$-\Delta v_n = \frac{\mu_n}{\|u_n\|_\infty} v_n - v_n^2 \qquad \text{in } B_{r/\epsilon_n}(0) \qquad \text{for all } n \geq 1. \tag{6.10}$$

Moreover, thanks to (5.10), it is apparent that

$$-\lambda|\Omega \setminus \Omega_0| < \frac{\mu_n}{\|u_n\|_\infty} \leq 0, \qquad n \geq 1.$$

Thus, we can asume, without loss of generality, that the next limit exists

$$\eta = \lim_{n\to\infty} \frac{\mu_n}{\|u_n\|_\infty} \in [-\lambda|\Omega \setminus \Omega_0|, 0]\,. \tag{6.11}$$

Since $r/\epsilon_n \to \infty$ as $n \to \infty$, and $0 < v_n \leq 1$ for all $n \geq 1$, by elliptic regularity and a standard compactness argument, there exists a strong solution of

$$-\Delta v = \eta v - v^2 \qquad \text{in } \mathbb{R}^N, \qquad 0 \leq v \leq 1,$$

denoted by v, such that, along some subsequence, labeled again by n, we have that $v_n \to v$ in $\mathcal{C}^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to \infty$. We claim that $v = 0$. Although this is a direct consequence of Du and Ma [6, Th. 1.1], by the sake of completeness, we will provide a very short self-contained proof of this fact. On the contrary, suppose $v > 0$ and let w denote the maximal solution of

$$-\Delta w = \eta w - w^2 \qquad \text{in } \mathbb{R}^N, \qquad 0 \leq w \leq 1. \tag{6.12}$$

As $\bar{w} := 1$ is a radially symmetric strict supersolution of (6.12), w must be radially symmetric. The function $\psi(r) = w(x)$, $r = |x|$, satisfies

$$\psi(r) > 0 \quad \text{and} \quad \psi'(r) = -\int_0^r \left(\frac{s}{r}\right)^{N-1} [\eta\psi(s) - \psi^p(s)]\, ds > 0.$$

Thus, ψ is increasing and, hence, the next limit is well defined

$$0 < \ell := \lim_{r\to\infty} \psi(r) \leq 1.$$

Necessarily, $\psi'(r) \to 0$ as $r \to \infty$ and $\psi''(r) \leq 0$ for sufficiently large r, which contradicts

$$-\psi''(r) - \frac{N-1}{r}\psi'(r) = \eta\psi(r) - \psi^p(r) < 0 \qquad \forall\, r \geq 0.$$

Therefore, $v = 0$ and hence, $\tilde{u}_n(x_{n,K}) = v_n(0) \to 0$ as $n \to \infty$, which entails $\max_K \tilde{u}_n \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \int_{\Omega\setminus\Omega_0} \tilde{u}_n = 0. \tag{6.13}$$

For each $n \geq 1$, let $x_n \in \bar{\Omega}$ be such that $u_n(x_n) = \|u_n\|_\infty$. By the proof of Theorem 5.1, we already know that $x_n \in \Omega_0$ for all $n \geq 1$. Thus, there exists some subsequence, relabeled by n, and $x_\infty \in \bar{\Omega}_0$ such that $\lim_{n\to\infty} x_n = x_\infty$. As we are assuming that $\bar{\Omega}_0 \subset \Omega$, there exists $r > 0$ such that $\bar{B}_r(x_n) \subset \Omega$ for all $n \geq 1$. Setting

$$z := (x - x_n)/\epsilon_n, \qquad w_n(z) := \tilde{u}_n(x), \qquad n \geq 1,$$

we find from (6.8) that

$$-\Delta w_n = \frac{\mu_n}{\|u_n\|_\infty} w_n + \lambda V(x) \left(\int_{\Omega\setminus\Omega_0} \tilde{u}_n \right) w_n - w_n^2 \qquad \text{in } B_{r/\epsilon_n}(0) \tag{6.14}$$

for all $n \geq 1$. Invoking to (6.11) and (6.13), by elliptic regularity and compactness, there exists a strong solution of

$$-\Delta w = \eta w - w^2 \qquad \text{in } \mathbb{R}^N,$$

say w, such that, along some subsequence, labeled again by n, we have that $w_n \to w$ in $\mathcal{C}^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$. Necessarily, $w = 0$, because $\eta \leq 0$ and $0 \leq w \leq 1$. Thus, $1 = \tilde{u}_n(x_n) = w_n(0) \to 0$ as $n \to \infty$, which is impossible. This contradiction together with Theorem 6.1 complete the proof of the theorem. □

Note that

$$\frac{2}{\sqrt{|\Omega_0||\Omega \setminus \Omega_0|}} \leq \lambda^* = \frac{|\Omega|}{|\Omega_0||\Omega \setminus \Omega_0|}$$

and that this inequality is always strict, unless $|\Omega_0| = |\Omega \setminus \Omega_0|$. Consequently, combining Theorem 2.1 with Theorem 6.2, it is apparent that, for the one-dimensional model,

$$\frac{2}{\sqrt{|\Omega_0||\Omega \setminus \Omega_0|}} \leq \lambda_c(2) \leq \lambda^* = \frac{|\Omega|}{|\Omega_0||\Omega \setminus \Omega_0|}.$$

Figure 6.1 shows a plot of $\mathfrak{C}_+$ for $\lambda \sim 4.9223 > \lambda^* \sim 2.0833$ when $\Omega = (-1, 1)$ and $\Omega_0 = (-0.6, 0.6)$. According to Theorem 6.1, it bifurcates sub-critically from $u = 0$.

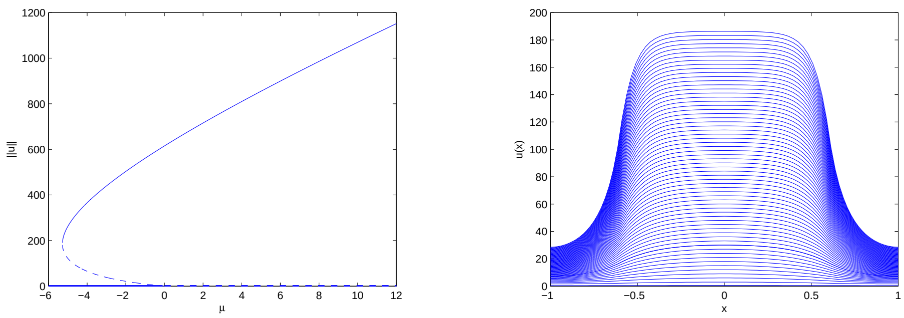

FIG. 6.1. $\mathfrak{C}_+$ *for* $\lambda \sim 4.922323718 > \lambda^*$ *and* $N = 1$ *and plots of a series of positive solutions along it.*

In the bifurcation diagram of Figure 6.1, as well as in all subsequent ones, stable solutions are represented with continuous lines, while unstable solutions are plotted with dash ones. The zero solution is stable for $\mu < 0$ and it becomes unstable as μ crosses 0 for any $\mu > 0$.

By the exchange stability principle of [4], the solutions bifurcating sub-critically from $u = 0$ must be unstable, with one-dimensional unstable manifold, until they reach the subcritical turning point on $\mathfrak{C}_+$, where they become stable for all further values of μ. Figure 6.2 shows the computed $\mathfrak{C}_+$ for $\lambda = 1.9 < \lambda^*$. By Theorem 6.1, it bifurcates super-critically from $u = 0$. These numerical experiments confirm that $\lambda_c(2) = \lambda^*$ for the choice of Section 3.

Combining the identity (6.3), or equivalently (6.7), with (6.5), reveals a new feature of a great significance. Namely,

$$\mu_2 \begin{cases} > 0 & \text{if} \quad |\Omega_0| > |\Omega \setminus \Omega_0|, \\ < 0 & \text{if} \quad |\Omega_0| < |\Omega \setminus \Omega_0|. \end{cases} \tag{6.15}$$

Consequently, the next local bifurcation result holds.

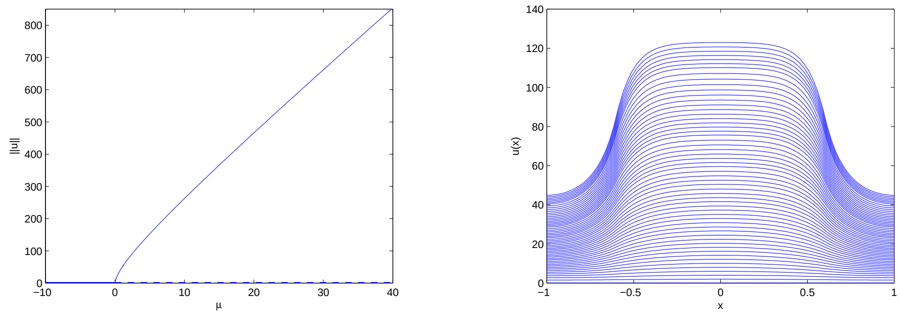

FIG. 6.2. *The component $\mathfrak{C}_+$ for $\lambda \sim 1.9 < \lambda^*$ in case $N = 1$ and plots of a series of positive solutions along it.*

THEOREM 6.3. *Suppose $|\Omega_0| > |\Omega \setminus \Omega_0|$. Then, there exist $\epsilon > 0$ and a continuous map $u : J_+ := [\lambda^*, \lambda^* + \epsilon) \to E$ such that $u(\lambda^*) = 0$ and $(\lambda, u(\lambda))$ is a positive solution of* (1.5) *for each $\lambda \in J_+$.*

Similarly, under condition $|\Omega_0| < |\Omega \setminus \Omega_0|$, there exist $\epsilon > 0$ and a continuous map $u : J_- := (\lambda^* - \epsilon, \lambda^*] \to E$ such that $u(\lambda^*) = 0$ and $(\lambda, u(\lambda))$ is a positive solution of (1.5) *for each $\lambda \in J_-$.*

Proof. Suppose $|\Omega_0| > |\Omega \setminus \Omega_0|$. Then, $\mu_2 > 0$. Consequently, by Theorem 6.1, the set of non-trivial solutions (μ, u) of (5.1) at $\lambda = \lambda^*$ in a neighbourhood of $(\mu, u) = (0, 0)$ consists of a quadratic supercritical turning point. Subsequently, it is convenient to emphasize the dependence on the parameter λ of the operator $\mathfrak{F}$ defined in (5.3), by setting $\mathfrak{F}(\lambda, \mu, u)$, rather than $\mathfrak{F}(\mu, u)$. Then, according to the proof of the main theorem of Crandall and Rabinowitz [3], for every (λ, μ, y, s) in a neighbourhood of $(\lambda^*, 0, 0, 0)$ in the product space $\mathbb{R} \times \mathbb{R} \times Y \times \mathbb{R}$, we can define

$$\mathfrak{G}(\lambda, \mu, y, s) := \begin{cases} s^{-1}\mathfrak{F}(\lambda, \mu, s(1 + y)), & s \neq 0, \\ D_u\mathfrak{F}(\lambda, \mu, 0)(1 + y), & s = 0. \end{cases} \tag{6.16}$$

As $\mathfrak{F}$ is real analytic, $\mathfrak{G}$ also is real analytic. Moreover,

$$\mathfrak{G}(\lambda, \mu, y, 0) = D_u\mathfrak{F}(\lambda, \mu, 0)(1 + y) = 1 + y - (\mu + 1)(-\Delta + 1)^{-1}(1 + y)$$

for all $y \in Y$ and $(\lambda, \mu) \sim (\lambda^*, 0)$ in $\mathbb{R}^2$. Thus,

$$\mathfrak{G}(\lambda, 0, 0, 0) = 1 - (-\Delta + 1)^{-1}1 = 1 - 1 = 0$$

and

$$D_{(\mu,y)}\mathfrak{G}(\lambda,0,0,0)(\mu,y) = y - (-\Delta+1)^{-1}y - \mu$$

for all $\lambda \in \mathbb{R}$ and $(\mu,y) \in \mathbb{R}\times Y$. Since $1 \notin R[-\Delta]$, it is easy to realize that

$$D_{(\mu,y)}\mathfrak{G}(\lambda,0,0,0) \in \mathrm{Iso}(\mathbb{R}\times Y; E). \tag{6.17}$$

Thus, due to the implicit function theorem, there exist $\epsilon > 0$ and a real analytic map

$$(\mu,y) : (-\epsilon,\epsilon)\times(\lambda^*-\epsilon,\lambda^*+\epsilon) \to \mathbb{R}\times Y$$

such that $(\mu(0,\lambda^*), y(0,\lambda^*)) = (0,0)$ and $\mathfrak{G}(\lambda,\mu(s,\lambda),y(s,\lambda),s) = 0$ for all $(s,\lambda) \in (-\epsilon,\epsilon)\times(\lambda^*-\epsilon,\lambda^*+\epsilon)$, which implies

$$\mathfrak{F}(\lambda,\mu(s,\lambda),s(1+y(s,\lambda))) = 0.$$

Actually, these are the unique zeros of $\mathfrak{G}$ in a neighborhood of $(\lambda,\mu,y,s) = (\lambda^*,0,0,0)$.

As for $\lambda \sim \lambda^*$ the set $\mathfrak{G}^{-1}(0)$ is a supercritical turning point in the parameter μ and, owing to (6.17), the linearization of $\mathfrak{G}$ along $\mathfrak{G}^{-1}(0)$ has maximal rank, applying the main theorem of [12, Section 6.10], also $\mathfrak{G}^{-1}(0)$ must be a supercritical quadratic turning point for $\lambda \sim \lambda^*$. As $\mu_1 < 0$ for $\lambda > \lambda^*$, necessarily for this range of values of λ, $\mathfrak{G}$ must have a non-trivial zero at $\mu = 0$, say $u(\lambda)$, such that $u(\lambda^*) = 0$. The continuity of this map is an easy consequence of the local structure of the solution set. This ends the proof of the first part, which can be easily adapted to cover the case $|\Omega_0| < |\Omega\setminus\Omega_0|$. □

As a byproduct from Theorem 6.3, $\lambda_c(2) < \lambda^*$ for the one-dimensional symmetric problem if $L > (b-a)/4$. Thus, by the existence of a priori bounds, the model admits at least two positive solutions for each $\lambda \in (\lambda_c(2),\lambda^*)$ and at least one for $\lambda = \lambda_c(2)$. By the invariance of the degree and the exchange stability principle, due to the existence of a priori bounds and thanks to Theorem 6.3, when $N \geq 2$ and $|\Omega_0| < |\Omega\setminus\Omega_0|$, there exists $R > 0$ such that (1.5) possesses at least two solutions for each $\lambda \in [\lambda^*-R,\lambda^*)$. However, this does not necessarily entail the existence of $\lambda_c(p,N) \leq \lambda^*-R$ for which (1.5) admits a positive solution if and only if $\lambda \geq \lambda_c(p,N)$, unless $N = 1$ and the problem is radially symmetric. Actually, in the general case when $N \geq 2$ it is an open problem to ascertain whether or not the set of λ's for which (1.5) has a positive solution is an interval. As when $p > 2$ perturbs from $p = 2$, for each $\lambda \in (0,\lambda^*)$ the problem (1.5) admits a further positive solution perturbing from $u = 0$ (see [10]), there exists an interval of λ's for which (1.5) has, at least, three positive solutions provided $|\Omega_0| < |\Omega\setminus\Omega_0|$ and $p > 2$, $p \sim 2$.

Figure 6.3 shows the *λ–bifurcation diagram* of the one-dimensional model studied in Section 2 in the special case when $\Omega_0 = (-0.1, 0.1)$, where $|\Omega_0| = 0.2 < |\Omega\setminus\Omega_0| = 1.8$ ($L = 0.9 > 0.5$). As established by Theorem 6.3, the solutions emanate sub-critically from $u = 0$ at $\lambda^* = |\Omega|/(|\Omega_0||\Omega\setminus\Omega_0|) = 5.55556$; actually, the path-following solver gave solutions of amplitude 10^{-6} from $\lambda = 5.45194$. All solutions along the bifurcated branch are unstable with one-dimensional unstable manifold until $\lambda = 3.73546$, where the diagram exhibits a supercritical turning point. For all further values of λ the solutions were stable. According to the numerics, it seems that $\lambda_c(2) \sim 3.73546$. Naturally, $\lambda_c(2) < \lambda^*$ and, according to Theorem 6.1, indeed $\lambda_c(2) \sim 3.73546 \geq 2(|\Omega_0||\Omega\setminus\Omega_0|)^{-1/2} \sim 3.33334$. Consequently, the estimates established by Theorem 6.1 seem very sharp. A rather striking feature is the fact that the positive solutions of the model are always increasing as we run along the bifurcation diagram. Hence, the solutions decrease with λ if they are unstable, while they grow if they are stable. This paradoxical behavior has not a counterpart in the

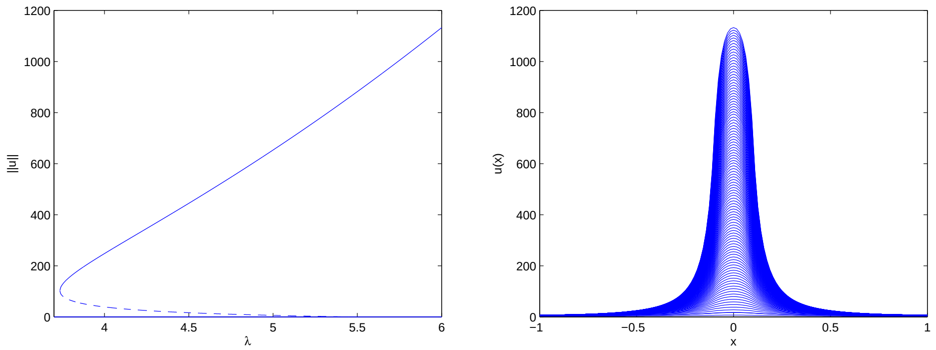

FIG. 6.3. *Bifurcation diagram for* $p = 2$ *with* $\Omega_0 = (-0.1, 0.1)$ *and plots of a series of positive solutions along it.*

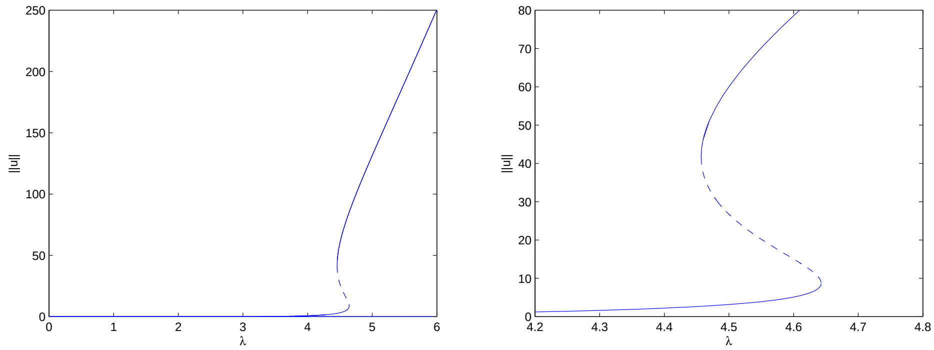

FIG. 6.4. *Global bifurcation diagram for* $p = 2.1$ *with* $\Omega_0 = (-0.1, 0.1)$ *and a magnification.*

context of superlinear indefinite parabolic equations, and might be utterly attributable to the lack of the maximum principle for non-local problems.

As predicted before, when p increases from $p = 2$, the trivial solution perturbs into a further positive solution. So, providing us with some interval of λ's where the model admits three solutions. Figure 6.4 shows two plots, at different scales, of the global bifurcation diagram computed for $p = 2.1$ with $\Omega_0 = (-0.1, 0.1)$. According to Theorem 4.1, the unique bifurcation value to positive solutions from $u = 0$ is $\lambda = 0$ and we indeed detected solutions of amplitude 10^{-6} from $\lambda \sim 0.01$. As the bifurcation diagram exhibits two turning points at $\lambda \sim 4.6426$ and $\lambda \sim 4.4571$, it is apparent that the problem possesses at least three positive solutions for every $\lambda \in (4.4571, 4.6426)$. Among them, two linearly stable and one unstable with one-dimensional unstable manifold. Figure 6.5 shows the profiles of a representative series of positive solutions along the bifurcation diagram of Figure 6.4. The positive solution always increase as we run along the bifurcation diagram starting at $\lambda = 0$. Consequently, they decay everywhere with λ if they are unstable, while they are increasing if they are stable.

7. Discussion. In this paper we have studied a variant of a logistic equation in the form of a nonlocal PDE which arises from crop raiding from large-bodied mammals living in the

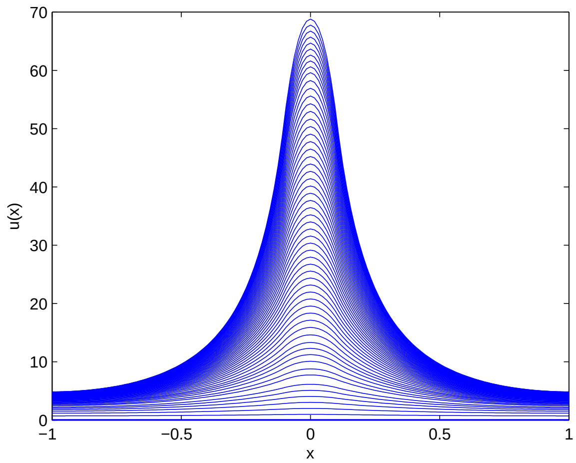

FIG. 6.5. *A series of plots of positive solutions along the bifurcation diagram of Figure 6.4.*

biodiversity-rich tropics. The well-known logistic equation can be written as

$$\frac{u'}{u} = r - \frac{u}{K},$$

where $u = u(t)$ is the population density, r is the intrinsic growth rate and K is the carrying capacity. In 1973 Gilpin and Ayala [9] introduced the θ-logistic equation

$$\frac{u'}{u} = r - \left(\frac{u}{K}\right)^{\theta}.$$

When $\theta = 1$, it is exactly the logistic equation, where the per capita growth rate is a linear function of population density. If $\theta > 1$, the increase in density from lower values has little effect on the per capita growth rate. It is only when density is much larger that an increase in density decreases substantially the per capita growth rate. This is the so called "crowding effect". When the environment is heterogeneous and the individual of the species are assumed to disperse randomly with a constant rate $d > 0$, the θ-logistic equation can be expressed as

$$\begin{cases} u_t = d\Delta u + ru - bu^p & \text{in } \Omega, \quad t > 0, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where $p \equiv \theta + 1 \geq 2$ and $b \equiv K^{-\theta}$. The change of variable $u = KU$ transforms (7.1) in

$$\begin{cases} U_t = d\Delta U + rU - U^p & \text{in } \Omega, \quad t > 0, \\ \frac{\partial U}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0, \\ U(\cdot, 0) = U_0 \equiv u_0/K & \text{in } \Omega. \end{cases} \tag{7.2}$$

According to Hutson et al. [11, Le. 2.2], for every $r > 0$ the constant $U^* \equiv r^{\frac{1}{p-1}} > 0$ is the unique positive steady state of (7.2) and $\lim_{t\to\infty} u(x,t) = U^*$ in $\mathcal{C}(\bar{\Omega})$.

In this paper, instead of (7.2), we consider (1.1), where the intrinsic growth rate is $\gamma V(x)(\int_{\Omega\setminus\Omega_0} U(y,t)\,dy)$, which is positive only on the forest region Ω_0 and is proportional to the "effort" to search for the food in the region of orchard or farm, $\Omega \setminus \Omega_0$. The effort is expressed as the total population in the region of orchard or farm. When $p = 2$, the per capita growth rate is linear in U. Our analysis shows that there is a bifurcation value

$$\gamma^* \equiv d^{\frac{p-2}{p-1}} \lambda^* = d^{\frac{p-2}{p-1}} \frac{|\Omega|}{|\Omega_0||\Omega \setminus \Omega_0|}$$

from $U = 0$ and that (7.2) possesses a positive steady state for all $\gamma > \gamma^*$. Moreover,

$$\begin{aligned} \frac{d}{dt} \int_\Omega U &= \gamma \left(\int_{\Omega_0} U \right) \left(\int_{\Omega \setminus \Omega_0} U \right) - \int_\Omega U^2 \\ &\leq \lambda \sqrt{|\Omega_0|} \left(\int_{\Omega_0} U^2 \right)^{\frac{1}{2}} \sqrt{|\Omega \setminus \Omega_0|} \left(\int_{\Omega \setminus \Omega_0} U^2 \right)^{\frac{1}{2}} - \int_{\Omega_0} U^2 - \int_{\Omega \setminus \Omega_0} U^2 \leq 0 \end{aligned}$$

provided

$$\gamma \leq \frac{2}{\sqrt{|\Omega_0|} \sqrt{|\Omega \setminus \Omega_0|}}.$$

Hence, if the population density $U(x, t)$ on Ω is small, then the species goes to extinct. On the other hand, if γ is large then the species survives. When $p > 2$, we have crowding effect when the population density is large; however when the population density is small, there is little effect of crowding compared to the growth. Hence, the species survives for any intrinsic growth rate γ.

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