International Journal of Bifurcation and Chaos, Vol. 21, No. 5 (2011) 1439–1456

DOI: 10.1142/S0218127411029136

# THE SPECTRUM OF CHAOTIC TIME SERIES (I): FOURIER ANALYSIS

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Received April 6, 2010; Revised October 13, 2010

The question of spectral analysis for deterministic chaos is not well understood in the literature. In this paper, using iterates of chaotic interval maps as time series, we analyze the mathematical properties of the Fourier series of these iterates. The key idea is the connection between the total variation and the topological entropy of the iterates of the interval map, from where special properties of the Fourier coefficients are obtained. Various examples are given to illustrate the applications of the main theorems.

*Keywords*: Li–Yorke chaos; topological entropy; total variation; Sobolev spaces; Fourier coefficients.

## 1. Introduction

The analysis of spectrum of a given function is important in the understanding of function behavior. Spectral analysis decomposes a function into a superposition of components, each with a special spatial and/or temporal frequency. Such a decomposition often reveals a certain pattern of the frequency distribution. For example, the so-called

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"noise" is a function or process whose spectral decomposition has prominent, irregularly distributed *high frequency components*. Spectral decomposition is a reversible process since by the inverse transform or superposition we recover the original function. Thus, no information is lost from spectral decomposition. Superposition can be discrete or continuous. When the domain (e.g. an interval) is finite, the spectrum of a function usually quantizes and is discrete. The Fourier transform or Fourier series expansion may be the most basic method which constitutes the foundation of all other spectral decomposition methods.

What kind of Fourier-spectral properties can we expect to have for a chaotic phenomenon? This is the topic we wish to address in this article. Here we are talking about deterministic chaos, which is an asymptotic time series represented by the iterates of a so-called chaotic interval map according to the definitions given in [Banks *et al.*, 1992; Block & Coppel, 1992; Devaney, 1989]. This topic is of obvious interest to many researchers. For instance, conducting a Google search by inputting "Fourier series of chaos," we obtain well over 1000 items. However, the greatest majority of such items study "Fourier series" and "chaos" disjointly. The rest of them are mostly on numerical simulation in nature. *Very few analytical results concerning the Fourier spectrum of chaotic time series exist so far, to the best of our knowledge.*

Interval maps as mentioned in [Block & Coppel, 1992; Devaney, 1989] above are advantageous for doing a mathematical analysis as chaos generated by them is comparatively simple and is quite well understood. There are various ways to characterize this type of chaos: *positivity of Lyapunov exponents or that of topological entropy*, *existence of homoclinic orbits or periodic points of specific order*, etc. Here, our main tool is the total variation of a function. It is known [Misiurewicz, 2004] that the topological entropy of a given interval map is positive (and, thus, the map is chaotic) if and only if the total variations of iterates of the map grow exponentially. This, together with certain fundamental properties of Fourier series related to the total variation of a function, enables us to obtain the desired interconnection between Fourier spectrum and chaos phenomena. Other properties ensue from the usual $L^p$ properties and the topological conjugacy. Basically, this paper may be viewed as *a study of chaos for interval maps from an integration point of view*, versus, say the Lyapunov exponent approach which is a differentiation approach. Nevertheless, we must clarify that we are *not* trying to determine the onset of chaos from the Fourier spectrum of the map $f$, which may constitute a futile attempt as the occurrence of chaos is *very sensitive* with respect to the profile of $f$ and, therefore, will also be very sensitive with respect to the Fourier coefficients of $f$ alone.

To be more precise and provide more heuristics, let $I = [0,1]$ be the unit interval and $f : I \to I$ be continuous (called an interval map). Denote $f^{ⓝ} = f \circ f \circ f \circ \cdots \circ f$: the $n$th iterative composition (or $n$th iterate) of $f$ with itself. Then the series $f, f^{②}, f^{③}, \ldots, f^{ⓝ}, \ldots,$ constitutes the time series we referred to in the above. Let us look at the profiles of two different cases, the first of which is nonchaotic and the second chaotic, as displayed in Figs. 1 and 2.

Our main question in this paper is: Can we analytically capture the highly oscillatory behavior of $f^{ⓝ}$ for a chaotic map $f$ through the Fourier series of $f^{ⓝ}$ when $n$ grows very large?

The organization of this paper is as follows. In Sec. 2, we provide a recap of prerequisite facts regarding interval maps and Fourier series. In Sec. 3, we offer three main theorems concerning Fourier series and chaos. Miscellaneous examples as applications are given in Sec. 4. A brief summary concludes the paper as in Sec. 5.

This paper is Part I of a series. In Part II, we will discuss wavelet analysis of chaotic interval maps.

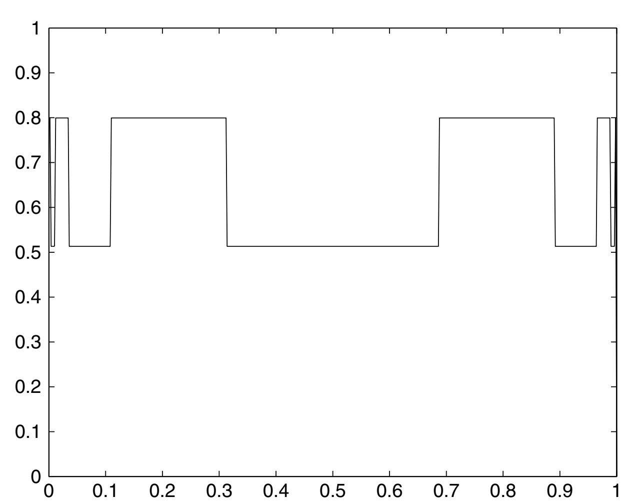


Fig. 1. The profile $f_\mu^{(100)}$ on $I$, where $f_\mu(x) = \mu x(1-x)$ is the quadratic map (cf. Example 3.2), here with $\mu = 3.2$, a case known to be nonchaotic. Note the nearly piecewise constant feature of the graph.

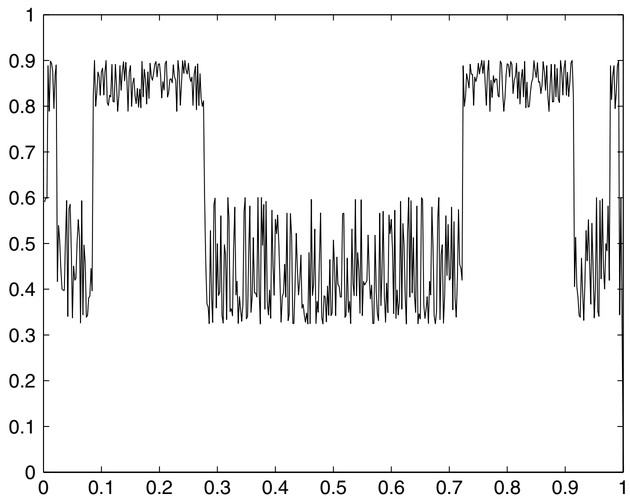


Fig. 2. The profile of $f_\mu^{100}$ on $I$, where $f_\mu$ is again the quadratic map, here with $\mu = 3.6$, a case known to be chaotic. Note that there are many oscillations in the graph, causing the total variations of $f_\mu^{n}$ to grow exponentially with respect to $n$ and, thus, we call it the occurrence of chaotic oscillations (or rapid fluctuations [Huang *et al.*, 2006]).

## 2. Recapitulation of Facts About Interval Maps and Fourier Series

This section recalls a brief summary of results needed for subsequent sections. For interval maps and chaos, we refer to the books by Devaney [1989] and Robinson [1999]. For Fourier series, cf. Edwards [1979].

The concept of topological entropy, introduced first by Ader *et al.* [1965] and studied by Bowen [1973, 1970, 1971] is a useful indicator for the complexity of system behavior. Let $X$ be a metric space with metric $d$. For $S \subset X$, define

$$d_{n,f}(x,y) = \sup_{0 \le j < n} d(f^{j}(x), f^{j}(y)); \quad x, y \in S.$$

We say that $S$ is $(n,\varepsilon)$-separated for $f$ if $d_{n,f}(x,y) > \varepsilon$ for all $x, y \in S, x \neq y$. We use $r(n,\varepsilon,f)$ to denote the largest cardinality of any $(n,\varepsilon)$-separated subset $S$ of $X$.

**Definition 2.1.** Let $(X,d)$ be a metric space and $f : X \to X$ be continuous. For any $\varepsilon$, the entropy of $f$ for a given $\varepsilon$ is defined by

$$h(\varepsilon, f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \ln(r(n,\varepsilon,f)).$$

The topological entropy of $f$ on $X$ is defined by

$$h(f) = \lim_{\varepsilon \downarrow 0} h(\varepsilon, f).$$

Let $X$ be a metric space and $f : X \to X$ be continuous. The nonwandering set of $f$ (see, e.g. [Zhou, 1997, p. 6]) $\Omega(f)$ is an invariant subset of $X$. We have the following.

**Theorem 1** (Proposition 8, [Zhou, 1997]). *Let $f : X \to X$ be a continuous map on a compact metric space $X$. Let $\Omega \subset X$ be the nonwandering set of $f$. Then the topological entropy of $f$ equals the entropy of $f$ restricted to $\Omega, h(f) = h(f|_\Omega)$.*

**Theorem 2.** *Let $f : I \to I$ be an interval map. Then the following conditions are equivalent:*

(1) *$f$ has a periodic point of period not being a power of 2.*
(2) *$f$ is strictly turbulent, i.e. there exist two compact subintervals $J$ and $K$ of $I$ with $J \cap K = \phi$ and a positive integer $k$ such that*
$$f^{k}(J) \cap f^{k}(K) \supset J \cup K.$$
(3) *$f$ has positive topological entropy.*
(4) *$f$ has a homoclinic point.*
(5) *$f$ is chaotic in the sense of Li–Yorke on the nonwandering set $\Omega(f)$ of $f$. i.e. there exists an uncountable set $S$ contained in $\Omega(f)$ such that*
   (a) $\limsup_{n\to\infty} d(f^{n}(x), f^{n}(y)) > 0 \ \forall x, y, x \neq y, \in S.$
   (b) $\liminf_{n\to\infty} d(f^{n}(x), f^{n}(y)) = 0 \ \forall x, y, \in S.$

Let $f : I \to I$ be a chaotic interval map. Then for many examples, the graphs of the iterates $f^{n} = f \circ f \circ \cdots \circ f$ ($n$ times), $n = 1, 2, 3, \ldots,$ exhibit very oscillatory behavior. The more so when $n$ grows. A useful way to quantify the oscillatory behavior is through the use of *total variations* (cf. [Chen *et al.*, 2004, p. 2164]). For any function $f$ defined on $I$, we let $V_I(f)$ denote the total variation of $f$ on $I$. If $V_I(f)$ is finite, we say that $f$ is a function of bounded variation.

We define the following function spaces:

$BV(I,I)$: the set of all functions of bounded variation mapping from $I$ to $I$;

$$W^{k,p}(I) = \left\{ f \in \mathcal{D}'(I) \,|\, \|f\|_{k,p} = \left[ \sum_{j=0}^{k} \int_I |f^{(j)}(x)|^p dx \right]^{1/p} < \infty \right\},$$

for $k = 0, 1, 2, \ldots, 1 \le p \le \infty$, where $\mathcal{D}'(I)$ is the space of distributions on $I$ and $f^{(k)}$ is the $k$th order distributional derivative of $f$. The case $p = \infty$ is interpreted in the sense of supremum a.e.

$\mathcal{F}_1$: the set of all functions $f \in C^0(I, I)$ such that $f^{ⓝ} \in BV(I, I)$ for $n \in \mathbb{N}$

$\mathcal{F}_2$: the set of all functions $f \in C^0(I, I)$ such that $f$ has finitely many extremal points.

It is clear that $\mathcal{F}_2 \subset \mathcal{F}_1$ and $W^{1,\infty}(I, I) \subset \mathcal{F}_1$.

**Theorem 3** [Misiurewicz & Szlenk, 1980]. *Let $f \in \mathcal{F}_1$. If $f$ satisfies any conditions in (1)–(5) in Theorem 2, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln(V_I(f^{ⓝ})) > 0, \tag{1}$$

*i.e. $V_I(f^{ⓝ})$ grows exponentially with respect to $n$. The converse is also true provided that $f \in \mathcal{F}_2$. Indeed, for $f \in \mathcal{F}_2$, we have*

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \ln(V_I(f^{ⓝ})).$$

The above is an outstanding theorem giving the connections between chaos, topological entropy and the exponential growth of total variations of iterates.

Let $f \in C^0(I, I)$. The map $f$ is called *topologically mixing* if, for every pair of nonempty open sets $U$ and $V$ of $I$, there exists a positive integer $N$ such that $f^{ⓝ}(U) \cap V \neq \phi$ for all $n > N$. And $f$ is said to have *sensitive dependence on initial condition* on a subinterval $J \subset I$ if there exists a $\delta > 0$, called a sensitive constant, such that for every $x \in J$ and every open set $U$ containing $x$, there exist a point $y \in U$ and a positive integer $n$ such that $|f^{ⓝ}(x) - f^{ⓝ}(y)| > \delta$.

**Theorem 4** [Chen *et al.*, 2004; Huang *et al.*, 2005]. *Assume that $f \in \mathcal{F}_1$. Then*

(1) *If $f$ is topologically mixing, then $V_J(f^{ⓝ})$ grows exponentially as $n \to \infty$ for any subinterval $J \subset I$;*
(2) *If $f$ has sensitive dependence on initial data, then $V_J(f^{ⓝ})$ grows unbounded as $n \to \infty$ for any subinterval $J \subset I$. The converse is also true provided that $f \in \mathcal{F}_2$.*
(3) *If $f \in \mathcal{F}_2$ and has sensitive dependence on initial data, then $V_I(f^{ⓝ})$ grows exponentially as $n \to \infty$;*
(4) *If $f$ has a periodic point of period four, then $V_I(f^{ⓝ})$ grows unbounded as $n \to \infty$.*

From Theorems 3 and 4, we know that the growth rates of the total variation of $f^{ⓝ}$ is strongly related to the dynamical complexity of $f$. The faster the total variations $V_I(f^{ⓝ})$ grow, the more fluctuations the graphs of $f^{ⓝ}$ have. This motivates us to define the following.

**Definition 2.2.** Let $f \in \mathcal{F}_1$. The map $f$ is said to have chaotic oscillations (or rapid fluctuation [Huang *et al.*, 2006]) if $V_I(f^{ⓝ})$ grows exponentially with respect to $n$, i.e. (1) holds.

Obviously, if $f \in \mathcal{F}_2$ has chaotic oscillations, then from Theorem 3 it follows that $h(f) > 0$. Thus $f$ is chaos in the sense of both Li–Yorke and Devaney [Li, 1993] and so $f$ satisfies the definition of chaos given in [Block & Coppel, 1992; Devaney, 1989].

Next, we recall some results about Fourier series. Let $f \in L^1(I)$. Denote

$$c_k = \int_0^1 e^{-2\pi ikx} f(x)dx, \quad k \in \mathbb{Z}. \tag{2}$$

The Fourier series of $f$ is defined to be

$$S(f)(\xi) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ik\xi}, \quad \xi \in \mathbb{R}. \tag{3}$$

The following is known to be true.

**Theorem 5.** *Let $f \in L^1(I)$ and let $c_k$ be defined by (2). Then*

(1) $\lim_{|k|\to\infty} c_k = 0$, *(the Riemann–Lebesgue Lemma).*
(2) *If $f$ is differentiable at $\xi_0$, then $S(f)(\xi_0) = f(\xi_0)$ in the sense that*

$$\lim_{M,N\to\infty} \sum_{k=-M}^{N} c_k e^{2\pi ik\xi_0} = f(\xi_0),$$

*(Dirichlet's Theorem).*

(3) *If $f \in \mathcal{F}_1$, then*

$$2\pi|kc_k| \leq 1 + V_I(f), \quad k \in \mathbb{Z}.$$

*Proof.* Here we only need to prove (3).

If $k = 0$, (3) holds obviously. Assume $k \neq 0$, then by (2), we have

$$c_k = \int_0^1 f(t)d\left[\frac{e^{-i2\pi kt}}{-i2\pi k}\right].$$

Set

$$g(t) = \frac{e^{-i2\pi kt}}{-i2\pi k}.$$

From the definition of an integral, for any given $\varepsilon > 0$, there exists a sufficiently fine partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval $[0, 1]$, such that

$$\left| c_k - \sum_{k=1}^{m} f(t_k)[g(t_k) - g(t_{k-1})] \right| < \varepsilon.$$

Denoting by $\sum$ the sum appearing within the absolute value signs above, and applying partial summation, we obtain

$$\begin{aligned} \sum &= f(1)g(1) - f(t_1)g(0) \\ &\quad - \sum_{k=1}^{m-1} [f(t_{k+1}) - f(t_k)]g(t_k). \end{aligned}$$

Thus,

$$\begin{aligned} c_k &< \varepsilon + |f(1) - f(t_1)||g(0)| \\ &\quad + \sum_{k=1}^{m-1} |f(t_{k+1}) - f(t_k)||g(t_k)| \\ &\le \varepsilon + \frac{1}{2\pi|k|} + \frac{1}{2\pi} V(f) \frac{1}{|k|}, \end{aligned}$$

since $f(x) \in [0, 1], \forall\, x \in [0, 1]$, $g(1) = g(0) = 1/(-i2\pi k)$ and $|g(t)| \le 1/(2\pi|k|)$. Letting $\varepsilon \to 0$, we have obtained the desired result. ■

## 3. Main Theorems on the Fourier Spectrum of Chaotic Time Series

Let $f \in \mathcal{F}_1$ and $f^{ⓝ}$ be the $n$th iterates of $f$. Denote

$$c_k^n(f) = \int_0^1 e^{-2\pi i k x} f^{ⓝ}(x)dx. \tag{4}$$

These numbers $c_k^n(f)$ contain the complex magnitude and phase information of the Fourier spectrum of the time series $f^{ⓝ}$, $n = 1, 2, 3, \ldots$. Extensive numerical simulations by using the fast Fourier transforms were performed by Roque-Sol [2006]; see the good collection of graphics therein. Those graphics manifest a basic pattern that when $f$ is a chaotic interval map, $|c_k^n(f)|$ have *spikes* when $n$ and $k$ are somehow related, as $n$ and $k$ both grow large.

Nevertheless, those numerical simulations do not offer concrete analytical results, as *aliasing* effect significantly degrades numerical accuracy when the frequency ($k$ in (4)) is high, in any Fourier transforms. Also, Fourier components can only be computed up to, say $k = O(10^6)$, on a laptop, with uncertain accuracies.

Therefore, mathematical analysis is imperative in order to determine the spectral relation $c_k^n$ between $n$ and $k$.

**Definition 3.1.** Let $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$. We say that $\phi$ grows exponentially if

$$\lim_{n\to\infty} \frac{1}{n} \ln|\phi(n)| \ge \alpha > 0, \quad \text{for some } \alpha.$$

**Main Theorem 1.** *Let $f \in \mathcal{F}_1$, and $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be an integer-valued function growing exponentially, such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln[|\phi(n) c_{\pm\phi(n)}^n(f)|] > 0.$$

*Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] \ge \alpha' > 0, \quad \text{for some } \alpha' > 0.$$

*Consequently, $f$ has chaotic oscillations.*

*Proof.* Since $f \in \mathcal{F}_1$, $f^{ⓝ}$ has bounded variation for any positive integer $n$. Applying Theorem 5(3) to $f^{ⓝ}$, we have

$$\begin{aligned} 2\pi|k c_k^n(f)| &\le |f^{ⓝ}(1) - f^{ⓝ}(0)| + V_I(f^{ⓝ}), \\ &\qquad \forall\, k = \pm 1, \pm 2, \ldots. \end{aligned}$$

Now, let $|k| = \phi(n)$. Then, noting that $|f^{ⓝ}(1) - f^{ⓝ}(0)| \le 2$, we have

$$2 + V_I(f^{ⓝ}) \ge 2\pi|\phi(n) c_{\phi(n)}^n(f)|,$$

implying

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] &\ge \lim_{n\to\infty} \frac{1}{n} \ln[|\phi(n) c_{\phi(n)}^n(f)|] \\ &= \alpha' > 0, \quad \text{for some } \alpha'. \end{aligned}$$

(Here, without loss of generality, we assume that $f$ is onto. Thus $\{V_I(f^{ⓝ})\}$ is increasing and the limit of $(1/n)\ln[V_I(f^{ⓝ})$ exists as $n$ tends to infinite.) Therefore the proof is complete. ■

*Remark 3.1*

(1) If $f \in \mathcal{F}_2$ satisfies the assumptions in Main Theorem 1, then $f$ has positive entropy by Theorem 3.
(2) The assumptions in Main Theorem 1 are *not necessary conditions* for $f$ to have chaotic

oscillations. For instance, consider the map

$$f(x) = \begin{cases} 3x & \text{if } 0 \le x < \dfrac{1}{3}, \\ 1 & \text{if } \dfrac{1}{3} \le x < \dfrac{2}{3}, \\ -3(x-1) & \text{if } \dfrac{2}{3} \le x \le 1. \end{cases}$$

Then $f \in \mathcal{F}_1$. A simple computation shows that

$$V_{[0,1]}(f^{ⓝ}) = 2^n.$$

So $f$ has chaotic oscillations. But the corresponding Fourier coefficient $c_k^n(f)$ satisfies

$$|c_k^n(f)| \le \int_I |f^{ⓝ}(x)|dx \le \left(\frac{2}{3}\right)^{n-1} \to 0,$$

as $n \to \infty$.

As an application of Main Theorem 1, we consider the tent map $T_m(x)$ defined by

$$T_m(x) = \begin{cases} mx, & 0 \le x < \dfrac{1}{m}, \\ \dfrac{m}{1-m}(x-1), & \dfrac{1}{m} \le x \le 1. \end{cases} \tag{5}$$

Here, choose $m = 2$ so we have a full tent map $T_2(x)$, symmetric with respect to $x = 1/2$. We have the following.

**Theorem 6.** *For the (full) tent map $T_2(x)$ (with $m = 2$ in (5)), we have the Fourier coefficients $c_k^n(T_2)$ given by*

$$c_k^n(T_2) = \begin{cases} -\dfrac{1}{\pi^2 s^2}, & \text{if } k = s2^{n-1}, \quad s = 1, 3, 5 \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* See Appendix A. ■

**Example 3.1.** For the full tent map $T_2(x)$, if we choose

$$|k| = |s|2^{n-1} \equiv \phi(n), \quad s = 1, 3, 5 \ldots,$$

then by Theorem 6, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln[|\phi(n)c^n_{\pm\phi(n)}(T_2)|] &= \lim_{n\to\infty} \frac{1}{n} \ln\left[s2^{n-1}\frac{1}{\pi^2 s^2}\right] \\ &= \lim_{n\to\infty} \frac{1}{n} \ln(2^{n-1}) \\ &= \ln(2) > 0. \end{aligned}$$

Thus, Main Theorem 1 applies, and $T_2(x)$ has chaotic oscillations.

*Remark 3.2.* From the proof of our Main Theorem 1, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_1(f^{ⓝ})] \ge \lim_{n\to\infty} \frac{1}{n} \ln[|\phi(n)c^n_{\phi(n)}(f)|]. \tag{6}$$

On the other hand, for the full tent map $T_2(x)$, we have from Example 3.1

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \ln[V_I(T_2^{ⓝ})] \\ &\qquad = \lim_{n\to\infty} \frac{1}{n} \ln[|\phi(n)c^n_{\phi(n)}(T_2)|] = \ln 2. \end{aligned}$$

Thus inequality (6) is quite tight.

The following corollary follows easily from Main Theorem 1.

**Corollary 3.1.** *Under the assumption that $f \in \mathcal{F}_1$ and that $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ grows exponentially, and*

$$|c^n_{\pm\phi(n)}(f)| \ge \beta > 0 \quad \textit{for } n \textit{ sufficiently large,}$$

*we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] \ge \alpha' > 0 \quad \textit{for some } \alpha'.$$

**Corollary 3.2.** *Let $f \in \mathcal{F}_1$. If $V_I(f^{ⓝ})$ remains bounded with respect to $n$, then*

$$\lim_{k\to\infty} |c_k^n(f)| = 0, \tag{7}$$

*uniformly for $n$.*

*Proof.* It follows from (3) in Theorem 5. ■

We know from (2) in Theorem 4 that $f$ is not chaotic in the sense of Devaney if $f \in \mathcal{F}_2$ and $V_I(f^{ⓝ})$ remains bounded with respect to $n$.

Nevertheless, (7) is weaker than the condition

$$\lim_{(n,k)\to\infty} |c_k^n(f)| = 0. \tag{8}$$

In fact, the boundedness of $V_I(f^{ⓝ})$ with respect to $n$ does not imply (8) in general. For instance, $f(x) = x$, $x \in [0,1] = I$, then $f^{ⓝ} = f$ and $V_I(f^{ⓝ}) = 1$ for any $n \in \mathbb{N}$, but $c_k^n(f) = c_k^1(f) \ne 0$, so (8) is violated.

**Example 3.2.** For the quadratic map $f_\mu \equiv \mu x(1-x), x \in I$, when $1 \le \mu \le 3$, we can prove by the similar approach in the proof of Lemma 2.3 in

[Huang, 2003] that $V_I(f_\mu^{ⓝ})$ remains bounded for all $n \in \mathbb{N}$. Thus, Corollary 3.2 applies.

A somewhat generalized version of Main Theorem 1 may be given as follows.

**Theorem 7.** *Let $f \in \mathcal{F}_1$ and there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln[\phi(n)] \equiv \alpha > 0.$$

*If*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left[(\phi(n)) \sum_{k\in\mathbb{Z}} |c_k^n(f)|^2 \sin^2\left(\frac{k\pi}{2\phi(n)}\right)\right] > 0, \tag{9}$$

*then*

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] = \alpha' > 0 \quad \textit{for some } \alpha'. \tag{10}$$

*In particular, if*

$$\sum_{k\in\mathbb{Z}} |c_k^n(f)|^2 \sin^2\left(\frac{k\pi}{2\phi(n)}\right) > 1,$$

*then (9) holds and so does (10).*

The proof of this theorem can be obtained from the following lemma by setting $r = \phi(n)$ and $g = f^{ⓝ}$ therein.

**Lemma 1** (Exercise 8.13, [Edwards, 1979]). *Suppose $g \in L^2(I)$. Then*

$$8r \sum_{k\in\mathbb{Z}} |c_k^1(g)|^2 \sin^2\left(\frac{k\pi}{2r}\right) \le \Omega_\infty g\left(\frac{\pi}{r}\right) V_I(g),$$

*for any positive number $r$, where*

$$\Omega_\infty g(a) = \sup_{0\le\delta\le a} \|T_\delta(g) - g\|_{C^0}, \quad (T_\delta g)(x) \equiv g(x-\delta),$$

*and $g$ is extended outside $I$ by periodic extension.*

In most cases, it is quite impossible to calculate the Fourier coefficients explicitly for $f^{ⓝ}$ for a *general* interval map $f$. In the following, we derive a sufficient condition so that we do not need to compute the Fourier coefficients directly. Instead, we need some conditions on the derivative of the map.

**Main Theorem 2.** *Let $f \in W^{1,\infty}(I, I)$ satisfy*

$$|f'|_{L^\infty(I)} = \gamma > 0.$$

*If*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] - \ln\gamma > 0, \tag{11}$$

*then $f$ has chaotic oscillations. Furthermore, if $f \in \mathcal{F}_2$, then $f$ has positive topological entropy and consequently, $f$ is chaotic in the sense of Li–Yorke.*

*Proof.* We have

$$2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right)^{1/2} = \left[\int_I |f^{ⓝ\prime}(x)|^2 dx\right]^{1/2},$$

where "prime" denotes the weak derivative of a given function. If $|f'|_{L^\infty(I)} = \gamma$, then we have

$$f^{ⓝ\prime}(x) = f'(f^{n-1}(x))f'(f^{n-2}(x)) \cdots f'(f(x))f'(x)$$

a.e. on $I$, and thus

$$|f^{ⓝ\prime}(x)| \le \gamma^n \quad \text{a.e. on } I.$$

We combine the above and now obtain

$$\begin{aligned}2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right)^{1/2} &= \left[\int_I |f^{ⓝ\prime}(x)||f^{ⓝ\prime}(x)|dx\right]^{1/2} \\ &\le \left[\int_I \gamma^n |f^{ⓝ\prime}(x)|dx\right]^{1/2} \\ &\le \gamma^{n/2}\left[\int_I |f^{ⓝ\prime}(x)|dx\right]^{1/2} \\ &\le \gamma^{n/2}[V_I(f^{ⓝ})]^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{n}\ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] &\le \frac{1}{n}\ln\left[\left(\frac{1}{2\pi}\right)^2 \gamma^n V_I(f^{ⓝ})\right] \\ &\le \ln(\gamma) + \frac{1}{n}\ln[V_I(f^{ⓝ})] \\ &\quad - \frac{2}{n}\ln(2\pi),\end{aligned}$$

where the last term vanishes as $n \to \infty$. By assumption, we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\ln[V_I(f^{ⓝ})] \ge \lim_{n\to\infty} \frac{1}{n}\ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] \\ - \ln(\gamma) > 0.\end{aligned}$$

Therefore $f$ has chaotic oscillations.

The second part of the theorem follows from Theorem 3. ■

**Example 3.3.** It follows from the proof of Main Theorem 2 that condition (11) is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}\ln\left[\int_I |f^{ⓝ\prime}(x)|^2 dx\right] - \ln(\gamma) > 0.$$

We consider the tent map $T_m(x)$ as given in (5) with $1 < m \leq 2$. It is easy to see that $V_I(T_m^{ⓝ}) = 2^n$ for any $m \in (1, 2]$. Now we prove that the tent map $T_m(x)$ satisfies the assumptions in Main Theorem 2. We have

$$\int_I |T_m^{ⓝ\prime}(x)|^2 dx = \left(\frac{m^2}{m-1}\right)^n. \tag{12}$$

(See Appendix B for the proof.)

For $\gamma$, we have

$$\gamma = |T'|_{L^\infty(I)} = \max\left(m, \frac{m}{m-1}\right) = \frac{m}{m-1},$$

since $1 < m \leq 2$. Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\ln\left[\sum_{k\in\mathbb{Z}} |kc_k^1(T_m)|^2\right] - \ln(\gamma)\\
&\quad= \lim_{n\to\infty}\frac{1}{n}\ln\int_0^1 |T_m^{ⓝ\prime}(x)|^2 dx - \ln(\gamma)\\
&\quad= [2\ln(m) - \ln(m-1)] - [\ln(m) - \ln(m-1)]\\
&\quad= \ln(m) > 0, \quad \forall\, 1 < m \leq 2.
\end{aligned}$$

**Example 3.4.** Another example is to consider the triangular map $H_q(x)$ defined by

$$H_q(x) = \begin{cases} qx & \text{if } 0 \leq x < \dfrac{1}{2},\\[2ex] q(1-x) & \text{if } \dfrac{1}{2} \leq x \leq 1,\end{cases}$$

where $0 < q \leq 2$. Figure 3 shows the graph of $H_q(x)$ with $1 < q \leq 2$.

In this case, coefficients $c_k^n(H_q)$ are extremely hard to evaluate. But since $\gamma = |H_q'|_{L^\infty} = q$, we have

$$\begin{aligned}
&\frac{1}{n}\ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] - \ln(\gamma)\\
&\quad= \frac{1}{n}\ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] - \ln(q)\\
&\quad= \frac{1}{n}\ln\left[\int_I |T^{ⓝ\prime}(x)|^2 dx\right] - \ln(q)
\end{aligned}$$

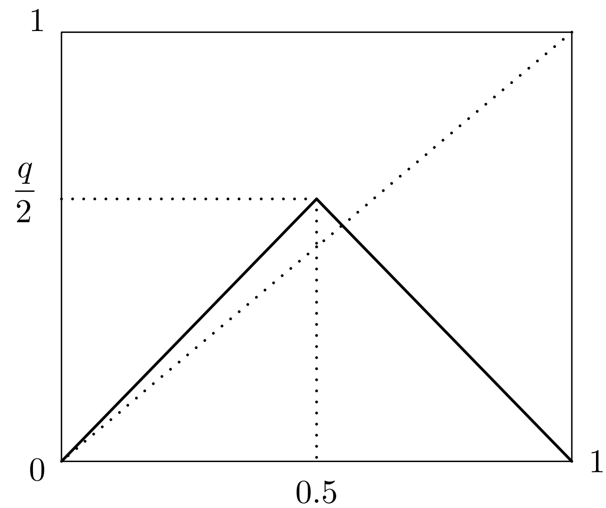


Fig. 3. The graph of the triangular map $H_q(x)$.

$$\begin{aligned}
&= \frac{1}{n}\ln(q^{2n}) - \ln(q)\\
&= 2\ln(q) - \ln(q) = \ln(q) > 0.
\end{aligned}$$

Thus the Triangular map $H_q(x)$ has positive entropy when $1 < q \leq 2$ by applying Main Theorem 2.

What we have given so far in this section are *sufficient conditions* for chaos. But for a given function $f \in W^{1,\infty}(I, I)$, there are some relations between $V_I(f)$ and $\|f\|_{W^{1,\infty}(I,I)}$, which will allow us to state some *necessary conditions*.

**Proposition 1.** *Let $f \in W^{1,\infty}(I, I)$. Then*

$$V_I(f^{ⓝ}) \leq 2\pi\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right]^{\frac{1}{2}}.$$

*Proof.* Let $f \in W^{1,\infty}(I, I)$ with the following Fourier series expansion

$$f(x) = \sum_{k\in\mathbb{Z}} c_k^1(f)e^{i2\pi kx}, \quad x \in I = [0, 1].$$

Then

$$\begin{aligned}
V_I(f) &= \int_0^1 |f'(x)|dx\\
&\leq \left(\int_0^1 dx\right)^{1/2}\left(\int_0^1 |f'(x)|^2 dx\right)^{1/2}
\end{aligned}$$

$$= \left(\int_0^1 dx\right)^{1/2}$$

$$\times \left(\int_0^1 |2i\pi \sum_{k\in\mathbb{Z}} kc_k^1(f)e^{i2\pi kx}|^2 dx\right)^{1/2}$$

$$\leq 2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^1(f)e^{i2\pi kx}|^2\right)^{1/2}$$

$$= 2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^1(f)|^2\right)^{1/2}.$$

Consequently, for the case of the $n$th iterates $f^{ⓝ}$ of $f$, it follows that

$$V_I(f^{ⓝ}) \leq 2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right)^{1/2}. \qquad \blacksquare$$

Now, let us see how the Fourier coefficients of $f^{ⓝ}$ and $f^{ⓝ'}$ behave when $f$ has chaotic oscillations.

**Main Theorem 3**

(1) *If* $f \in W^{1,\infty}(I, I)$ *and* $f$ *has chaotic oscillations, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] > 0.$$

(2) *If* $f \in \mathcal{F}_2 \cap W^{1,\infty}(I)$ *and has positive entropy, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right] > 0.$$

*Proof.* For (1), since $f \in W^{1,\infty}(I, I)$ and $W^{1,\infty}(I, I) \subset \mathcal{F}_1$, for any positive $n$, $f^{ⓝ}$ has bounded total variation. Assume that $f$ has chaotic oscillations. Then

$$\frac{1}{n} \ln[V_I(f^{ⓝ})] \geq \alpha > 0,$$

for some $\alpha$. By Proposition 1, we have

$$2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right)^{1/2} \geq V_I(f^{ⓝ}).$$

Thus

$$\lim_{n\to\infty} \frac{1}{n} \ln\left[\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right]$$

$$= 2 \lim_{n\to\infty} \frac{1}{n} \ln\left[2\pi\left(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\right)^{1/2}\right]$$

$$\geq 2 \lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})]$$

$$= 2\alpha > 0.$$

(2) follows from (1) and Theorem 3. $\blacksquare$

We now consider the effects of topological conjugacy. Let $f : I \to I$, $g : J \to J$ be continuous maps on compact intervals. We say $f$ is topologically conjugate to $g$ if there exists a homeomorphism $h : I \to J$ such that

$$hf = gh. \tag{13}$$

Furthermore, if there exists a bi-Lipschitz $h : I \to J$ such that (13) holds, then we say $f$ is Lipschitz conjugate to $g$. Without loss of generality, we assume $J = I$.

Recall that $h : I \to J$ is said to be bi-Lipschitz if both $h$ and its inverse $h^{-1}$ are Lipschitz maps. Thus $f$ is topologically conjugate to $g$ if they are Lipschitz conjugate to each other.

Since topological entropy is an invariant of topological conjugacy, we have the following.

**Theorem 8.** *Let* $f : I \to I$ *and* $g : I \to I$ *belong to* $\mathcal{F}_2$. *If* $f$ *is topologically conjugate to* $g$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] = \lim_{n\to\infty} \frac{1}{n} \ln[V_I(g^{ⓝ})],$$

*in particular,* $f$ *has chaotic oscillations iff* $g$ *does.*

*Proof.* It follows from Theorem 3. $\blacksquare$

**Theorem 9.** *Let* $f : I \to I$ *and* $g : I \to I$ *belong to* $\mathcal{F}_1$. *If* $f$ *is Lipschitz conjugate to* $g$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] = \lim_{n\to\infty} \frac{1}{n} \ln[V_I(g^{ⓝ})],$$

*in particular,* $f$ *has chaotic oscillations iff* $g$ *does.*

*Proof.* The bi-Lipschitz property of $h$ implies that $h$ is strictly monotone. Assume that $h$ is strictly increasing. (If it is strictly decreasing, the proof is the same.) Thus, for any partition $x_1 < x_2 < \cdots < x_n$ on $I$, $h(x_1) < h(x_2) < \cdots < h(x_n)$ is also a partition on $I$. So, we have

$$\begin{aligned}\sum_{i=0}^{n-1}&|f(x_{i+1}) - f(x_i)| \\ &= \sum_{i=0}^{n-1} |h^{-1}(g(h(x_{i+1}))) - h^{-1}(g(h(x_i)))| \\ &\leq |h^{-1}|_{\text{Lip}} \sum_{i=0}^{n-1} |g(h(x_{i+1})) - g(h(x_i))| \\ &\leq |h^{-1}|_{\text{Lip}} V_I(g),\end{aligned}$$

where $|h^{-1}|_{\text{Lip}}$ is a Lipschitz constant of $h^{-1}$. Therefore,

$$V_I(f) \leq |h^{-1}|_{\text{Lip}} V_I(g).$$

By the same reasoning, we get

$$V_I(f^{ⓝ}) \leq |h^{-1}|_{\text{Lip}} V_I(g^{ⓝ}),$$

for any positive integer $n$. Thus

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] \leq \lim_{n\to\infty} \frac{1}{n} \ln[V_I(g^{ⓝ})].$$

Since $g^{ⓝ}$ is also Lipschitz conjugate to $f^{ⓝ}$ for any positive integer $n$, by the same argument, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(g^{ⓝ})] \leq \lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})].$$

The proof is complete. ■

**Example 3.5.** The quadratic map $f_4(x) = 4x(1 - x)$, where we choose $\mu = 4$ in $f_\mu$ (cf. Example 3.2) is known to be topologically conjugate to the full tent map $T_2$ (cf. Theorem 6):

$$f_4 = h \circ T_2 \circ h^{-1},$$

where

$$h(x) = \sin^2\left(\frac{\pi x}{2}\right),$$

$$h^{-1}(y) = \frac{2}{\pi} \sin^{-1} \sqrt{y}; \quad x, y \in [0, 1].$$

We have

$$h'(x) = \pi \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} = \frac{\pi}{2} \sin(\pi x) \in L^\infty(I),$$

$$(h^{-1})'(y) = \frac{1}{\pi} \frac{1}{\sqrt{y}} \frac{1}{\sqrt{1-y}} \in L^{2-\delta}(I), \quad \text{for any } \delta > 0.$$

Because $(h^{-1})'$ is not in $L^\infty(I)$, we see that $f_4$ and $T_2$ are *not* Lipschitz conjugate to each other. Therefore, Theorem 9 is *not* applicable. Nevertheless, we have $f_4, T_2 \in \mathcal{F}_2$, so Theorem 8 is applicable, and we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln V_I(f_4^{ⓝ}) = \lim_{n\to\infty} \ln V_I(T_2^{ⓝ}).$$

Since

$$V_I(T_2^{ⓝ}) = 2^n,$$

we have

$$\lim_{n\to\infty} \frac{1}{n} V_I(f_4^{ⓝ}) = \lim_{n\to\infty} \frac{1}{n} V_I(T_2^{ⓝ}) = \ln 2 > 0.$$

**Example 3.6.** The preceding Example 3.5 gives support to the importance of the assumption that $f, g \in \mathcal{F}_2$ in order for Theorem 8 to hold. Here we give an example that if $f \notin \mathcal{F}_2$, then it is possible to find a continuous function $g : I \to I$ such that $f$ and $g$ are topologically conjugate: $h \circ f \circ h^{-1} = g$, and

$$\lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] > 0, \quad \text{but} \quad \lim_{n\to\infty} \ln[V_I(g^{ⓝ})] = 0.$$

We define $f$ to be a *piecewise linear*, continuous function on $I$ satisfying

$$\begin{cases} f\left(\dfrac{1}{2^{2k}}\right) = \dfrac{1}{2^{2k}}, & k = 0, 1, 2, \ldots \\ f\left(\dfrac{1}{2^{2k+1}}\right) = 0, & k = 0, 1, 2, \ldots, f(0) = 0. \end{cases} \tag{14}$$

For this $F$, it is straightforward to verify the following properties:

$$\text{(i)}\ f\left(\frac{1}{4}x\right) = \frac{1}{4} f(x), \quad \text{for all } x \in [0, 1]; \tag{15}$$

$$\text{(ii)}\ f^{ⓝ}\left(\frac{1}{4}x\right) = \frac{1}{4} f^{ⓝ}(x), \quad \text{for } n = 1, 2, \ldots, \quad \text{for all } x \in [0, 1]; \tag{16}$$

(iii) for any $k = 1, 2, \ldots,$

$$V_{[0,\frac{1}{4^k}]}(f^{ⓝ}) = \frac{1}{4} V_{[0,\frac{1}{4^{k-1}}]}(f^{ⓝ}) = \cdots = \frac{1}{4^k} V_{[0,1]}(f^{ⓝ}). \tag{17}$$

Indeed, (ii) follows from (i) and (iii) follows from (ii). Thus, we have

$$
\begin{aligned}
V_{[0,1]}(f^{n}) &= \sum_{k=0}^{\infty} V_{[\frac{1}{2^{k+1}},\frac{1}{2^k}]}(f^{n}) \\
&= V_I(f^{n-1}) + 2\sum_{k=1}^{\infty} V_{[0,\frac{1}{4^k}]}(f^{n-1}) \\
&= V_I(f^{n-1}) + 2\sum_{k=1}^{\infty} \frac{1}{4^k} V_I(f^{n-1}) \\
&= \left[1 + 2\sum_{k=1}^{\infty}\frac{1}{4^k}\right] V_I(f^{n-1}) \\
&= \frac{5}{3} V_I(f^{n-1}).
\end{aligned}
$$

Therefore $V_I(f^{n}) = (5/3)^n$, and

$$
\lim_{n\to\infty} \frac{1}{n} \ln V_I(f^{n}) = \ln\left(\frac{5}{3}\right) > 0.
$$

We next set out to construct a strictly increasing map $h \in C^\infty(I, I)$ satisfying

$$
h(0) = 0, \quad h(1) = 1, \quad \text{and} \quad \frac{1}{n}\ln V_I(h\circ f^{n}) = 0.
$$

We need to determine the number of the local maxima of $f^{n}$. For $j = 1, 2, \ldots$, define

$$
\begin{aligned}
a_{j,n} = {} & \text{the number of points at which } f^{n} \text{ takes} \\
& \text{the local maximal value } \frac{1}{2^{2j}} = \frac{1}{4^j}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
a_{1,n} &= n; \\
a_{2,n} &= a_{1,n} + 2(a_{1,1} + a_{1,2} + \cdots + a_{1,n-1}); \qquad (18) \\
&= n + 2[1 + 2 + \cdots + (n-1)] = n^2.
\end{aligned}
$$

By induction, for $j > 1$, we have

$$
a_{j+1,n} = a_{j,n} + 2\sum_{k=1}^{n-1} a_{j,k}.
$$

Thus,

$$
V_I(f^{n}) = 1 + 2\left(\frac{a_{1,n}}{4} + \frac{a_{2,n}}{4^2} + \cdots + \frac{a_{j,n}}{4^j} + \cdots\right).
$$

We now show that

$$
a_{j,n} \le n^j.
$$

This is true for $j = 1$ by (18). Assume that

$$
a_{j,k} \le k^j \quad \text{for } j = 1, 2, \ldots, n.
$$

Then

$$
\begin{aligned}
a_{j+1,n} &= a_{j,n} + 2\sum_{k=1}^{n-1} a_{j,k} \\
&\le n^i + 2[1 + 2^j + \cdots + (n-1)^j] \\
&\le n^{j+1}.
\end{aligned}
$$

So induction is complete.

We define $h : I \to I$ by

$$
h(0) = 0, \quad h\left(\frac{1}{4^j}\right) = \frac{1}{4^{j^2+j}}, \quad j = 0, 1, 2, \ldots,
$$

and require that $h$ be $C^\infty$ and strictly increasing. Thus, we have

$$
\begin{aligned}
V_I(h\circ f^{n}) &= 1 + 2\Bigg(\frac{a_{1,n}}{4^{1+1}} + \frac{a_{2,n}}{4^{2^2+2}} \\
&\qquad + \cdots + \frac{a_{j,n}}{4^{j^2+j}} + \cdots\Bigg) \\
&\le 1 + 2\sum_{j=1}^{\infty}\left(\frac{n}{4^j}\right)\cdot\frac{1}{4^j} \\
&\le 1 + 2\sum_{j=1}^{[\frac{\ln n}{\ln 4}]+1}\left(\frac{n}{4^j}\right)^j + 2\sum_{j=1}^{\infty}\frac{1}{4^j} \\
&\le 3 + 2\sum_{j=1}^{[\frac{\ln n}{\ln 4}]+1}\left(\frac{n}{4}\right)^j \\
&= 1 + 2\left[\frac{1 - \left(\frac{n}{4}\right)^{[\frac{\ln n}{\ln 4}]+1}}{1 - \left(\frac{n}{4}\right)}\right] \\
&\le 3\cdot\left(\frac{n}{4}\right)^{[\frac{\ln n}{\ln 4}]+1}, \quad \text{for large } n.
\end{aligned}
$$

Hence

$$
\frac{1}{n}\ln V_I(h\circ f^{n}) \le \frac{\ln 3 + \left(\left[\frac{\ln n}{\ln 4}\right] + 1\right)\ln\left(\frac{n}{4}\right)}{n} \to 0 \qquad (19)
$$

as $n \to \infty$.

We define

$g = h \circ f \circ h^{-1}$, i.e. $g$ is topologically conjugate to $f$.

Since $h^{-1}$ is strictly increasing, we have

$$V_I(g^{ⓝ}) = V_I(h \circ f^{ⓝ} \circ h^{-1}) = V_I(h \circ f^{ⓝ}).$$

By (19), we have

$$\lim_{n\to\infty} \frac{1}{n} \ln V_I(g^{ⓝ}) = \lim_{h\to\infty} \frac{1}{n} V_I(h \circ f^{ⓝ}) = 0.$$

## 4. Miscellaneous Consequences

We offer some examples as additional applications of the theory in Sec. 3.

**Example 4.1** (Application to PDEs). Here, we show an application to the case of chaotic oscillations of the wave equation with a van der Pol nonlinear boundary conditions, as studied by Chen *et al.* [1998b], Huang [2003] and Huang *et al.* [2005]. Consider the wave equation

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad 0 < x < 1, \ \ t > 0, \quad (20)$$

with a nonlinear self-excitation (i.e. van der Pol) boundary condition at the right end $x = 1$:

$$w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \quad 0 \le \alpha \le 1, \ \ \beta > 0,$$

and a linear boundary condition at the left end $x = 0$:

$$w_t(0,t) = -\eta w_x(0,t), \quad \eta > 0, \ \ \eta \ne 1, \ \ t > 0.$$

The remaining two conditions we require are the initial conditions

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in [0,1].$$

Then using Riemann invariants

$$u = \frac{1}{2}(w_x + w_t),$$

$$v = \frac{1}{2}(w_x - w_t),$$

the above becomes a first order hyperbolic system

$$\frac{\partial}{\partial t}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix},$$

where at the boundary $x = 0$ and $x = 1$, the reflection relations take place

$$v(0,t) = \frac{1+\eta}{1-\eta} u(0,t) \equiv G(u(0,t)),$$

$$u(1,t) = F(v(1,t)),$$

where $F(x) \equiv x + g(x)$ and $g(x)$ is the unique solution to the cubic equation

$$\beta g^3(x) + (1-\alpha)g(x) + 2x = 0, \quad x \in \mathbb{R}.$$

Solutions $w_x(x,t), w_t(x,t)$ of the wave equation display chaotic oscillatory behavior if $G \circ F$ or equivalently $F \circ G$ is a *chaotic interval map*, when $\alpha, \beta, \eta$ lie in a certain region. We therefore deduce the following. Assume that for given $\alpha, \beta, \eta : 0 < \alpha \le 1, \beta > 0$ and $\eta > 0, \eta \ne 1$, the map $G \circ F$ is chaotic and that the initial conditions $w_0(\cdot)$ and $w_1(\cdot)$ satisfy $w_0, w_1 \in \mathcal{F}_2$, and

$$w_0 \in C^2([0,1]), \quad w_1 \in C^2([0,1]),$$

and that *compatibility conditions*

$$w_1(0) = -\eta w_0'(0),$$

$$w_0'(1) = \alpha w_1(1) \beta w_1^3(1),$$

$$w_0''(0) = -\eta w_1'(0),$$

$$w_1'(0) = [\alpha - 3\beta w_1^2(1)] w_0''(1),$$

are satisfied. Then there exist $A_1 > 0, A_2 > 0$ such that if

$$|w_0'|_{C^0(I)}, \quad |w_1|_{C^0(I)} \le A_1 \quad w_0' \ne 0 \quad \text{or} \quad w_1 \ne 0,$$

then

$$|w_x|_{C^0(I)}, \quad |w_t|_{C^0(I)} \le A_2.$$

In addition, we require that $u(x,0)$ and $v(x,0)$ take values in the "strange attractors" of $G \circ F$. Since $G \circ F$ and $F \circ G$ are chaotic, $V_I(u(\cdot,t))$ and $V_I(v(\cdot,t))$ grow exponentially with respect to $t$. Therefore, $V_I(w_x(\cdot,t))$ and $V_I(w_t(\cdot,t))$ also grow exponentially with respect to $t$. Since

$$\int_0^1 |w_{xx}(x,t)| dx = V_I(w_x(\cdot,t)),$$

$$\int_0^1 |w_{xt}(x,t)| dx = V_I(w_t(\cdot,t)),$$

and

$$\left[\int_0^1 |w_{xx}(x,t)|^2 dx\right]^{1/2} \ge \int_0^1 |w_{xx}(x,t)| dx,$$

$$\left[\int_0^1 |w_{xt}(x,t)|^2 dx\right]^{1/2} \ge \int_0^1 |w_{xt}(x,t)| dx,$$

we obtain the following exponential growth results (stated in logarithmic form)

$$\lim_{n\to\infty}\frac{1}{n}\ln\left[\int_0^1(|w_{xx}(x,n+t_0)| + |w_{xt}(x,n+t_0)|)dx\right] > 0,$$

$$\lim_{n\to\infty}\frac{1}{n}\ln\left[\int_0^1(|w_{xx}(x,n+t_0)|^2 + |w_{xt}(x,n+t_0)|^2)dx\right] > 0,$$

for any $t_0 > 0$.

The above *exponential growth results for a large class of initial states* are in strong contrast to the traditional "well-posedness" results where *uniform exponential boundedness in time* (with reference to initial conditions) are established. Note that these estimates have not been obtainable by any other methods (such as the energy multiplier method).

The chaotic behavior of the wave equation (20) with other boundary conditions have been also considered by Chen *et al.* [1998a, 1998c].

**Example 4.2** (Entropy and Hausdorff dimension). Let $X$ be a nonempty compact metric space and $f\colon X \to X$ a Lipschitz continuous map with Lipschitz constant $L$, that is, $\forall\, x, y \in X, f$ satisfies

$$d(f(x), f(y)) \leq Ld(x,y), \quad \text{where } d \text{ is the metric of } X.$$

The topological entropy $h(f, Y)$ of $f$ on an arbitrary subset $Y \subset X$, given by Bowen [1973], is much like the Hausdorff dimension, with the "size" of a set reflecting how $f$ acts on it. Let $\mathcal{A}$ be a finite open cover of $X$. For a set $B \subset X$ we write $B \prec \mathcal{A}$ if $B$ is contained in some element of $\mathcal{A}$.

Let $n_{f,\mathcal{A}}(B)$ be the largest non-negative integer such that $f^{ⓚ}(B) \prec \mathcal{A}$ for $k = 0, 1, 2, \ldots, n_{f,\mathcal{A}} - 1$. If $B \not\prec \mathcal{A}$ then $n_{f,\mathcal{A}}(B) = 0$, and if $f^k(B) \prec \mathcal{A}$ for all $k$ then $n_{f,\mathcal{A}}(B) = \infty$. Now, we define

$$\mathrm{diam}_{\mathcal{A}}(B) = \exp(-n_{f,\mathcal{A}}(B)),$$

and

$$D_{\mathcal{A}}(\mathcal{B}, \lambda) = \sum_{i=1}^{\infty}(\mathrm{diam}_{\mathcal{A}}(B_i))^{\lambda}$$

for any family $\mathcal{B} = \{B_i\}_1^{\infty}$ of subsets of $X$ and any $\lambda \in \mathbb{R}^+$. Define a measure $\mu_{\mathcal{A},\lambda}(Y)$ by

$$\mu_{\mathcal{A},\lambda}(Y) = \lim_{\varepsilon\to 0}\inf_{\mathcal{B}}\{D_{\mathcal{A}}(\mathcal{B},\lambda) : \mathcal{B} = \{B_i\}_1^{\infty}, \cup B_i \supseteq Y, \mathrm{diam}_{\mathcal{A}}(B_i) < \varepsilon\},$$

which has similar properties as the classical Hausdorff measure:

$$\mathcal{H}^{\lambda}(Y) = \lim_{\varepsilon\to 0}\inf\left\{\sum_i(\mathrm{diam}(B_i))^{\lambda} : \cup_i B_i \supseteq Y \text{ and } \sup_i\{|B_i|\} < \varepsilon\right\},$$

i.e. there exists $h(f, Y, \mathcal{A})$ such that

$$\mu_{\mathcal{A},\lambda}(Y) = \infty \quad \text{for } \lambda < h(f, Y, \mathcal{A}),$$

$$\mu_{\mathcal{A},\lambda}(Y) = 0 \quad \text{for } \lambda > h(f, Y, \mathcal{A}).$$

Finally, we define

$$h(f, Y) = \sup\{h(f, Y, \mathcal{A}) : \mathcal{A} \text{ is a finite open cover of } Y\}.$$

This number $h(f, Y)$ is the topological entropy of $f$ on the set $Y$. If $Y = X$, then by [Bowen, 1973, Proposition 1] we get

$$h(f, X) = h(f),$$

the topological entropy of $f$.

From Misiurewicz [2004], we have the following.

**Theorem 10** [Misiurewicz, 2004]. *For any $Y \subset X$, the Hausdorff dimension $\mathcal{H}_d(Y)$ of $Y$, for a Lipschitz continuous map $f$ with Lipschitz constant $L > 1$, satisfies the inequality*

$$\mathcal{H}_d(Y) \geq \frac{h(f, Y)}{\ln(L)}.$$

**Corollary 4.1.** *Under the same assumptions as in Theorem 10, the Hausdorff dimension of $X$ satisfies*

$$\mathcal{H}_d(X) \geq \frac{h(f, X)}{\ln(L)} = \frac{h(f)}{\ln(L)}, \quad L > 1.$$

*Remark 4.1.* More recently, Dai and Jiang [2006] generalized Theorem 10 to the case that the phase space $X$ is a metric space satisfying the second countability (but not necessarily compact).

Now, let us consider the case of an interval map $f : I \to I$ and let $\mathcal{L}$ the set defined by

$$\mathcal{L} = \{\text{all Lipschitz continuous functions } f\colon I \to I, \text{ with Lipschitz constant greater than } 1\}. \tag{21}$$

**Theorem 11.** *Let $f \in W^{1,\infty}(I) \cap \mathcal{F}_2 \cap \mathcal{L}$. Let $c_k^n$ be the kth Fourier coefficient of $f^{ⓝ}$. If $\Omega(f)$ denotes the set of nonwandering points of $f$, then*

$$\mathcal{H}_d(\Omega(f)) \geq \frac{1}{\ln(L)} \lim_{n\to\infty} \ln|kc_k^n|, \quad k = \pm 1, \pm 2, \ldots$$

*Proof.* Apply Theorem 10 to $Y = \Omega(f)$, which is an invariant, closed and therefore compact set, and obtain

$$\mathcal{H}_d(\Omega(f)) \geq \frac{h(f, \Omega(f))}{\ln(L)} = \frac{h(\Omega(f))}{\ln(L)} = \frac{h(f|_\Omega)}{\ln(L)} = \frac{h(f)}{\ln(L)}.$$

But we already know from (3) in Theorem 5 that

$$2 + V_I(f^{ⓝ}) \geq 2\pi|kc_k^n|.$$

Therefore

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \ln[V_I(f^{ⓝ})] \geq \lim_{n\to\infty} \frac{1}{n} \ln|kc_k^n|,$$

implying

$$\mathcal{H}_d(\Omega(f)) \geq \frac{1}{\ln(L)} \lim_{n\to\infty} \frac{1}{n} \ln|kc_k^n|. \qquad \blacksquare$$

**Corollary 4.2.** *Let $f$ be a function satisfying the hypotheses of Theorem 11, and let $\phi : \mathbb{N} \to \mathbb{N}$ grow exponentially (satisfying Definition 3.1), and*

$$\lim_{n\to\infty} |c_{\pm\phi(n)}^n| > 0.$$

*Then the Hausdorff dimension of the nonwandering set $\Omega(f)$ is positive, i.e.*

$$\mathcal{H}_d(\Omega(f)) > 0.$$

*Proof.* Since $\phi : \mathbb{N} \to \mathbb{N}$ grows exponentially, there are $\alpha_1 > 0, \alpha_2 > 0$ such that

$$\phi(n) \geq \alpha_1 e^{\alpha_2 n}.$$

By setting $k = \pm\phi(n)$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln|kc_k^n| &= \lim_{n\to\infty} \frac{1}{n} \ln|\phi(n) c_{\pm\phi(n)}^n| \\ &\geq \lim_{n\to\infty} \frac{1}{n} \ln|\alpha_1 e^{\alpha_2 n} c_{\pm\phi(n)}^n| \\ &= \alpha_2 + \lim_{n\to\infty} \frac{\ln(\alpha_1)}{n} \\ &\quad + \lim_{n\to\infty} \frac{\ln|c_{\pm\phi(n)}^n|}{n} \\ &= \alpha_2 > 0.\end{aligned}$$

Hence

$$\mathcal{H}_d(\Omega(f)) \geq \frac{1}{\ln(L)} \lim_{n\to\infty} \frac{1}{n} \ln|kc_k^n| \geq \frac{\alpha_2}{\ln(L)} > 0.$$

$\blacksquare$

In the case of the full tent map $T_2(x)$, the Lipschitz constant of $T_2(x)$ is obviously 2, i.e. $L = 2$.

Also, from Theorem 6 and Example 3.1, by choosing

$$\phi(n) = s2^{n-1}, \quad s = 1, 3, 5, \ldots, \tag{22}$$

we obtain

$$\mathcal{H}_d(\Omega(T)) \geq \frac{\ln(2)}{\ln L} = 1.$$

Therefore

$$\mathcal{H}_d(\Omega(T)) = 1.$$

Finally, we show an application of the Sturm–Hurwitz Theorem [Katriel, 2003], an important theorem in the oscillation theory of Fourier series, to the theory that we are developing here. Let $X$ be a closed subset of the interval $I = [0, 1]$ and $f : X \to X$ a continuous mapping. Let $Y \subset X$. Denote $\mathcal{J}$ the set of all possible subintervals of $I$, and for $\mathcal{J}|_Y$ the family of all subintervals of $I = [0, 1]$, each restricted to $Y$.

**Definition 4.1.** Let $Y \subset X$. A subset $\mathcal{A}$ of $\mathcal{J}|_Y$ is called a cover of $Y$ if $\mathcal{A}$ has finitely many elements and

$$Y \subset \cup_{A\in\mathcal{A}} A.$$

A cover $\mathcal{A}$ of $Y$ is called $f$-mono on $Y$ if for any $A \in \mathcal{A}$ the map $f|_A$ is monotone.

Using the above definition we can see piecewise monotone functions in a slightly different way, namely, the following.

**Definition 4.2.** A map $f$ is called piecewise monotone (p.m.), if there exists an $f$-mono cover of $X$.

**Definition 4.3.** Let $f$ be a p.m. continuous mapping from an interval $I$ into itself. Denote

$$l_n = \min\{\text{Card } \mathcal{A} : \mathcal{A} \text{ is an } f^{ⓝ}\text{-mono cover}\},$$
(Card means cardinality).

From [Misiurewicz & Szlenk, 1980], we have the following.

**Lemma 2.** *If $f : I \to I$ is a p.m. continuous map, then*

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \ln(l_n).$$

The Sturm–Hurwitz Theorem states that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous $2\pi$-periodic function and $d_k$ is the $k$th Fourier coefficient of $g$, i.e.

$$d_k = \int_0^{2\pi} g(x)e^{-ikx}dx, \quad k = 0, \pm 1, \pm 2, \ldots,$$

and if $d_{k_0}$ is the first nonzero Fourier coefficients of $g$, i.e.

$$d_k = \begin{cases} 0 & \text{if } |k| < k_0, \\ \neq 0 & \text{if } |k| = k_0, \end{cases}$$

then the function $g$ has at least $2k_0$ distinct zeros in the interval $[0, 2\pi]$.

**Theorem 12.** *Let $f \in C^0(I, I)$ be a p.m. mapping with $f(0) = f(1)$. If there exists a map $\phi : \mathbb{N} \to \mathbb{N}$ satisfying*

$$\ln[\phi(n)] \geq \alpha_1 + \alpha_2 n, \quad \textit{for some } \alpha_1 \in \mathbb{R}, \ \alpha_2 > 0,$$

*such that the $k$th Fourier coefficient of $f^{ⓝ}$ satisfies*

$$c_k^n = \begin{cases} 0 & \textit{if } |k| < \phi(n), \\ \neq 0 & \textit{for some } |k| = \phi(n), \end{cases}$$

*then*

$$h_{\text{top}}(f) > 0.$$

*Proof.* For a given $n \in \mathbb{N}$ define

$$g_n(x) = f^{ⓝ}\left(\frac{x}{2\pi}\right), \quad x \in [0, 2\pi],$$

then $g_n(0) = g_n(2\pi)$, so we can extend $g_n$ to the whole line $\mathbb{R}$ continuously with period $2\pi$. Applying the Sturm–Hurwitz Theorem, we have that $g_n(x)$ has at least $2\phi(n)$ zeros in the interval $[0, 2\pi]$. This implies that $f^{ⓝ}$ has at least $2\phi(n)$ distinct zeros in $[0, 1]$. Therefore

$$l_n \geq 2\phi(n).$$

It follows from Lemma 2 that

$$\begin{aligned} h_{\text{top}}(f) &= \lim_{n\to\infty} \frac{1}{n} \ln[l_n] \\ &\geq \lim_{n\to\infty} \frac{1}{n}[\ln[2] + \ln[\phi(n)]] \\ &\geq \lim_{n\to\infty} \frac{1}{n}[\ln[2] + \alpha_1 + \alpha_2 n] \\ &= \alpha_2 > 0. \end{aligned}$$

■

**Example 4.3.** Consider $T_2(x)$, the full tent map. We can apply Theorem 12 by using Theorem 6 and (22). The proof of Theorem 12 gives

$$h_{\text{top}}(T_2) \geq \ln 2.$$

Actually, $h_{\text{top}}(T_2) = \ln 2$.

## 5. Conclusions

Chaotic interval maps generate time series (constituted by their iterates) manifesting progressively oscillatory behavior. Such oscillations must be reflected in the high order Fourier coefficients of the time series but no quantitative, analytical results were known previously. Our work here was first motivated by numerical simulation in [Roque-Sol, 2006]. Later, we recognized some fundamental relation between coefficients of the Fourier series and the total variation of a given function (Theorem 5 (3)). Further, a concrete example (Theorem 6 and Example 3.1) was constructed. These have lead to a host of other related results and applications, yielding a form of "integration theory" for chaotic interval maps.

In Part II, we will continue the investigation by using wavelet transforms.

## Acknowledgments

Goong Chen is supported in part by the Texas Advanced Research Project (ARP) Grant No. 010336-0149-2009 from THECB and Qatar National Research Fund (QNRF) Grant No. NPRP 09-462-1-074. Sze-Bi is supported in part by grants from the National Science Council and the Ministry of Education of Republic of China. Yu Huang is supported in part by the National Natural Science Foundation of China (Grants 10771222 and 11071263) and the NSF of Guangdong Province.

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# Appendices

## A. Evaluation of the Fourier Coefficients of $T_2^{ⓝ}(x)$ for the Tent Map $T_2(x)$

Here we present the proof of Theorem 6.

The $n$th iterate of the Triangular map $T_2(x)$ is given by

$$T_2^{ⓝ}(x) = \begin{cases} 2^n x - 2(l-1), & \text{if } \dfrac{2(l-1)}{2^n} \le x \le \dfrac{2l-1}{2^n}, \\[2ex] -2^n x + 2l, & \text{if } \dfrac{2l-1}{2^n} \le x \le \dfrac{2l}{2^n}, \end{cases}$$

for $l = 1, 2, \ldots, 2^{n-1}$. Now,

$$\begin{aligned} c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^{ⓝ}(x) e^{-2\pi ikx} dx \\ &= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi ikx} dx \right. \\ &\quad \left. + \sum_{l=1}^{2^{n-1}} \int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}} [-2^n x + 2l] e^{-2\pi kx} dx \right\} \\ &= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi ikx} dx \right\} \\ &\quad + \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}} [-2^n x + 2l] e^{-2\pi ikx} dx \right\} \\ &\equiv I_1 + I_2, \end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi i k x} dx\right\} \\
&= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_0^{\frac{1}{2}} 2t e^{-2\pi i k \frac{t+(l-1)}{2^{n-1}}} \frac{dt}{2^{n-1}}\right\}; \\
&\qquad (2t = 2^n x - 2(l-1), dt = 2^{n-1} dx) \\
&= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t[e^{-\frac{i\pi k t}{2^{n-2}}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}}] dt\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}} dt\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} dt\right\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}} \\
&= \frac{1}{2^{n-1}}\left\{\int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} dt\right\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}} \\
&= \left\{\frac{2^{-2}}{-i\pi k} e^{-\frac{i\pi k}{2^{n-1}}} + \frac{2^{n-3}}{\pi^2 k^2}(e^{-\frac{i\pi k}{2^{n-1}}} - 1)\right\} \\
&\quad \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}} [-2^n x + 2l] e^{-2\pi i k x} dx\right\} \\
&= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \int_0^{\frac{1}{2}} 2t e^{-2\pi i k \frac{l-t}{2^{n-1}}} \frac{dt}{2^{n-1}}\right\}; \\
&\qquad (2t = -2^n x + 2l, dt = -2^{n-1} dx) \\
&= \frac{1}{2}\left\{\sum_{l=1}^{2^{n-1}} \frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t[e^{\frac{i\pi k t}{2^{n-2}}} e^{-\frac{i\pi k l}{2^{n-2}}}] dt\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{\frac{i\pi k t}{2^{n-2}}} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-1}}} dt\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{\frac{i\pi k t}{2^{n-2}}} dt\right\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= \frac{1}{2^{n-1}}\left\{\int_0^{\frac{1}{2}} t e^{\frac{i\pi k t}{2^{n-2}}} dt\right\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= \left\{\frac{2^{-2}}{i\pi k} e^{\frac{i\pi k}{2^{n-1}}} + \frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-1}}} - 1)\right\} \\
&\quad \times \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^{ⓝ}(x) e^{-2\pi i k x} dx \\
&= I_1 + I_2 \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1 - e^{-\frac{i\pi k}{2^{n-1}}})^2 \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}}.
\end{aligned}
$$

Now, if $k \neq s2^{n-1}, s = 1, 2, \ldots,$ then

$$
\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^{ⓝ}(x) e^{-2\pi i k x} dx = I_1 + I_2 \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1 - e^{-\frac{i\pi k}{2^{n-1}}})^2 \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(1 - e^{-i2\pi k})\left\{\frac{1 - e^{-\frac{i\pi k}{2^{n-1}}}}{1 + e^{-\frac{-i\pi k}{2^{n-1}}}}\right\} \\
&= 0.
\end{aligned}
$$

On the other hand, if $k = s2^{n-1}, s = 1, 3, 5, \ldots,$ then

$$
\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^{ⓝ}(x) e^{-2\pi i k x} dx = I_1 + I_2 \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1 - e^{-\frac{i\pi k}{2^{n-1}}})^2 \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= -\frac{1}{\pi^2 s^2}.
\end{aligned}
$$

## B. Some Calculations for $T_m^{ⓝ}(x)$ Needed for Example 3.3

Here we present the proof of (12).

Consider the tent map given by (5), i.e.

$$
T_m(x) = \begin{cases} mx, & 0 \leq x < \dfrac{1}{m}, \\[2ex] \dfrac{m}{1-m}(x-1), & \dfrac{1}{m} \leq x \leq 1, \end{cases}
$$

for $1 < m \leq 2$.

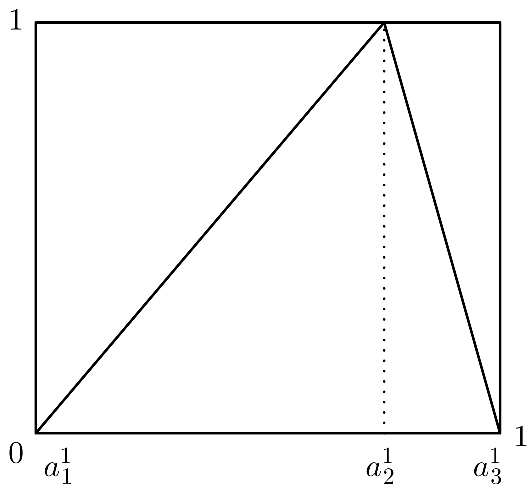


Fig. 4. The graph of the tent map $T_m$.

$T_m$ has an extremal point $x = a_2^1 = 1/m$, as displayed in Fig. 4.

After iterating $T_m$ twice, $T_m^{(2)}$ has three extremal points:

$$a_2^2 = a_1^1 + \frac{1}{m}(a_2^1 - a_1^1) = \frac{1}{m^2},$$

$$a_3^2 = a_2^1,$$

$$a_4^2 = a_3^1 + \frac{1}{m}(a_2^1 - a_3^1) = 1 + \frac{1}{m}\left(\frac{1}{m} - 1\right).$$

See Fig. 5.

After $n$ iterates, if we denote by $a_l^n, l = 2, 3, \ldots, 2^n$, the extremal points of $T_m^{(n)}$ and let $a_1^n = 0, a_{2^n+1}^n = 1$, be the two boundary points, then we have

$$a_{2l-1}^n = a_l^{n-1}, \quad l = 1, 2, \ldots, 2^{n-1} + 1,$$

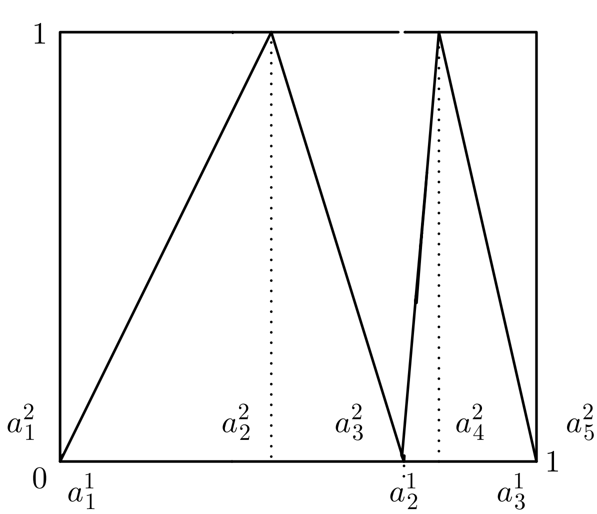


Fig. 5. The graph of the tent map $T_m^2$.

$$a_{2l}^n = \begin{cases} a_l^{n-1} + \dfrac{1}{m}(a_{l+1}^{n-1} - a_l^{n-1}), & \text{if } l \text{ is odd,} \\[2ex] a_{l+1}^{n-1} + \dfrac{1}{m}(a_l^{n-1} - a_{l+1}^{n-1}), & \text{if } l \text{ is even,} \end{cases}$$

for $l = 1, 2, \ldots, 2^{n-1}$.

Thus

$$a_{2l}^n - a_{2l-1}^n = \lambda\lambda^{b_l}(1-\lambda)^{(n-1)-b_l}, \tag{B.1}$$

$$a_{2l+1}^n - a_{2l}^n = (1-\lambda)\lambda^{b_l}(1-\lambda)^{(n-1)-b_l}, \tag{B.2}$$

where

$$\lambda = \frac{1}{m}, \quad 1 < m \le 2,$$

$l - 1 = c_{n-2}2^{n-2} + c_{n-3}2^{n-3} + \cdots + c_1 2^1 + c_0$, the binary expansion of $l - 1$, with $c_j = 0$ or $1$,

$b_l =$ number of zeroes in the binary coefficients $\{c_{n-2}, c_{n-3}, \ldots, c_1, c_0\}$.

On the other hand, we know that

$$T_m^{(n)'} = \frac{1}{a_{2l}^n - a_{2l-1}^n}, \qquad \text{on } (a_{2l-1}^n, a_{2l}^n),$$

$$T_m^{(n)'} = -\frac{1}{a_{2l+1}^n - a_{2l}^n}, \quad \text{on } (a_{2l}^n, a_{2l+1}^n).$$

It follows from (B.1) and (B.2) that

$$\begin{aligned} \int_0^1 |T_m^{(n')}|^2 dx &= \sum_{l=1}^{2^{n-1}} \left[ \int_{a_{2l-1}^n}^{a_{2l}^n} \frac{1}{(a_{2l}^n - a_{2l-1}^n)^2} dx + \int_{a_{2l}^n}^{a_{2l+1}^n} \frac{1}{(a_{2l+1}^n - a_{2l}^n)^2} dx \right] \\ &= \sum_{l=1}^{2^{n-1}} \left[ \frac{1}{a_{2l}^n - a_{2l-1}^n} + \frac{1}{a_{2l+1}^n - a_{2l}^n} \right] \\ &= \sum_{l=1}^{2^{n-1}} \frac{1}{\lambda^{b_l+1}(1-\lambda)^{n-b_l}} \\ &= \sum_{b=0}^{n-1} \binom{n-1}{b} \frac{1}{\lambda^{b+1}(1-\lambda)^{n-b}} \\ &= \left[\frac{1}{\lambda(1-\lambda)}\right]^n \\ &= \left(\frac{m^2}{m-1}\right)^n. \end{aligned}$$

Thus (12) holds.