Dynamics of A Two Complementary Resources Model with Internal Storage and Periodic Input

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1 Introduction

In this paper, we consider

$$\begin{cases} \frac{dS}{dt} = (S^{(0)}(t) - S)D - f_{S1}(S, Q_{S1})u_1 - f_{S2}(S, Q_{S2})u_2, \\ \frac{dR}{dt} = (R^{(0)}(t) - R)D - f_{R1}(R, Q_{R1})u_1 - f_{R2}(R, Q_{R2})u_2, \\ \frac{dQ_{Si}}{dt} = f_{Si}(S, Q_{Si}) - \min\{\mu_{Si}(Q_{Si}), \mu_{Ri}(Q_{Ri})\}Q_{Si},\ i = 1, 2, \\ \frac{dQ_{Ri}}{dt} = f_{Ri}(R, Q_{Ri}) - \min\{\mu_{Si}(Q_{Si}), \mu_{Ri}(Q_{Ri})\}Q_{Ri},\ i = 1, 2, \\ \frac{du_i}{dt} = [\min\{\mu_{Si}(Q_{Si}), \mu_{Ri}(Q_{Ri})\} - D]u_i,\ i = 1, 2, \end{cases} \tag{1.1}$$

where $S^{(0)}(t+\tau) = S^{(0)}(t) \geq 0$ and $R^{(0)}(t+\tau) = R^{(0)}(t) \geq 0$, for some period $\tau > 0$.

We assume that for each $i = 1, 2$ and $N = S,\ R$, $\mu_{Ni}(Q_{Ni})$ is defined and continuously differentiable for $Q_{Ni} \geq Q_{\min,Ni} > 0$ and satisfies

(H1) $\mu_{Ni}(Q_{Ni}) \geq 0$, $\mu'_{Ni}(Q_{Ni}) > 0$ and is continuous for $Q_{Ni} \geq Q_{\min,Ni}$, $\mu_{Ni}(Q_{\min,Ni}) = 0$.

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We assume that $f_{Ni}(N, Q_{Ni})$ is continuously differentiable for $N > 0$ and $Q_{Ni} \geq Q_{\min,Ni}$ and satisfies

$$f_{Ni}(0, Q_{Ni}) = 0, \ \frac{\partial f_{Ni}(N, Q_{Ni})}{\partial N} > 0, \ \frac{\partial f_{Ni}(N, Q_{Ni})}{\partial Q_{Ni}} \leq 0. \tag{H2}$$

It is better to assume that

$$f_{Ni}(0, Q_{Ni}) = 0, \ \frac{\partial f_{Ni}(N, Q_{Ni})}{\partial N} \geq 0, \ \frac{\partial f_{Ni}(N, Q_{Ni})}{\partial Q_{Ni}} \leq 0. \tag{H2}$$

In [4], the authors studied the homogeneous case of (1.1).

2 Single Population

In this section, we first consider the single population model. Mathematically, it simply means that we set $u_1 = 0$ or $u_2 = 0$ in equations (1.1). In order to simplify notation, all subscripts are dropped in the remaining equations and we consider

$$\begin{cases} \frac{dS}{dt} = (S^{(0)}(t) - S)D - f_S(S, Q_S)u, \\ \frac{dR}{dt} = (R^{(0)}(t) - R)D - f_R(R, Q_R)u, \\ \frac{dQ_S}{dt} = f_S(S, Q_S) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_S, \\ \frac{dQ_R}{dt} = f_R(R, Q_R) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_R, \\ \frac{du}{dt} = [\min\{\mu_S(Q_S), \mu_R(Q_R)\} - D]u, \end{cases} \tag{2.1}$$

with initial values in the domain

$$\mathbf{X} = \{(S, R, Q_S, Q_R, u) \in \mathbb{R}^5_+ : Q_S \geq Q_{\min,S}, \ Q_R \geq Q_{\min,R}\}. \tag{2.2}$$

It is easy to show that $\mathbf{X}$ is positively invariant for the system (2.1). Putting $u = 0$ in the first two equations of (2.1) results in

$$\frac{dS}{dt} = (S^{(0)}(t) - S)D, \tag{2.3}$$

and

$$\frac{dR}{dt} = (R^{(0)}(t) - R)D. \tag{2.4}$$

The linear equations (2.3) and (2.4) have unique globally attractive positive $\tau-$periodic solutions $S = S^*(t) = S^*(t+\tau)$ and $R = R^*(t) = R^*(t+\tau)$, respectively, which describe the available nutrient in a phytoplankton-free chemostat.

Now, putting $S = S^*(t)$ and $R = R^*(t)$ in system (2.1) results in

$$\begin{cases} \frac{dQ_S}{dt} = f_S(S^*(t), Q_S) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_S, \\ \frac{dQ_R}{dt} = f_R(R^*(t), Q_R) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_R, \end{cases} \tag{2.5}$$

with initial values in the domain

$$\Gamma = \{(Q_S, Q_R) \in \mathbb{R}_+^2 : Q_N \geq Q_{\min,N},\ N = S, R\}. \tag{2.6}$$

It is easy to show that Γ is positively invariant for the system (2.5).

Lemma 2.1. *The system* (2.5) *has a unique* $\tau-$*periodic solution* $(Q_S^*(t), Q_R^*(t))$ *to which all solutions are attracted.*

Proof. We first show that every solution of system (2.5) exists for all $t \geq 0$. Let $[0, \sigma)$ be the maximal interval of existence. We need to prove that $\sigma = +\infty$. By the continuation theorem, it suffices to prove that the solution is bounded on $[0, \sigma)$. Motivated by the proofs in [3, Proposition 3], we consider the dynamics of the variable

$$V(t) = \frac{1}{2}[(Q_S(t))^2 + (Q_R(t))^2].$$

Using the monotonicity of f_N, the inequality $Q_N \leq 1 + Q_N^2$ and the fact that $Q_N \geq Q_{\min,N}$, for $N = S, R$, it follows that $V(t)$ satisfies

$$\begin{aligned} \frac{dV(t)}{dt} &= \frac{dQ_S(t)}{dt}Q_S + \frac{dQ_R(t)}{dt}Q_R \\ &\leq f_S(S^*(t), Q_S)Q_S + f_R(R^*(t), Q_R)Q_R \\ &\leq f_S(S^*_{\max}, Q_{\min,S})[1 + Q_S^2] + f_R(R^*_{\max}, Q_{\min,R})[1 + Q_R^2] \\ &\leq f_S(S^*_{\max}, Q_{\min,S}) + f_R(R^*_{\max}, Q_{\min,R}) \\ &+ 2\max\{f_S(S^*_{\max}, Q_{\min,S}), f_R(R^*_{\max}, Q_{\min,R})\}V(t), \end{aligned} \tag{2.7}$$

where $S^*_{\max} := \max_{t\in[0,\tau]} S^*(t)$ and $R^*_{\max} := \max_{t\in[0,\tau]} R^*(t)$. The resulting inequality implies that $V(t)$ are bounded in finite time intervals. So are $Q_S(t)$ and $Q_R(t)$.

Next, we show that every solution of system (2.5) is bounded eventually. From the first equation of (2.5), it follows that

$$\frac{dQ_S}{dt} \geq f_S(S^*_{\min}, Q_S) - \mu_S(Q_S)Q_S,$$

where $S^*_{\min} := \min_{t\in[0,\tau]} S^*(t)$. The resulting inequality and the fact that every solution of system (2.5) exists globally, imply that $\liminf_{t\to\infty} Q_S(t) \geq Q^0_S$, where Q^0_S is the unique root of $f_S(S^*_{\min}, Q_S) - \mu_S(Q_S)Q_S = 0$, and hence, $Q^0_S > Q_{\min,S}$. Similarly, it follows from the second equation of (2.5) that there exists a unique constant $Q^0_R > Q_{\min,R}$ such that $\liminf_{t\to\infty} Q_R(t) \geq Q^0_R > Q_{\min,R}$. From the above discussions and (H1), it follows that there exists $\delta_0 > 0$ and $T > 0$ such that

$$\min\{\mu_S(Q_S(t)), \mu_R(Q_R(t))\} \geq \delta_0, \ \forall\, t \geq T.$$

From (2.5), it follows that

$$\begin{cases} \frac{dQ_S}{dt} \leq f_S(S^*_{\max}, Q_S) - \delta_0 Q_S, \ \forall\, t \geq T, \\ \frac{dQ_R}{dt} \leq f_R(R^*_{\max}, Q_R) - \delta_0 Q_R, \ \forall\, t \geq T, \end{cases}$$

where $S^*_{\max}$ and $R^*_{\max}$ were defined in the previous discussion. Thus, every solution of system (2.5) is bounded eventually.

System (2.5) is a planar, competitive, periodic system of differential equations. The main results for such system is that all bounded solutions are asymptotic to a periodic solution (See Chap. 7, Sect.4 in [9]). Let $(Q^*_S(t), Q^*_R(t))$ be a periodic solution of system (2.5). Then (H1) and (H2) imply that $Q^*_S(t) \geq Q_{\min,S}$ and $Q^*_R \geq Q_{\min,R}$ for all $t \geq 0$, and hence, $(Q^*_S(t), Q^*_R(t))$ is a positive periodic solution of system (2.5). If one can prove that the positive periodic solution $(Q^*_S(t), Q^*_R(t))$ of system (2.5) is unique, then the proof is done. Suppose for contradiction that $(Q^{1*}_S(t), Q^{1*}_R(t))$ and $(Q^{2*}_S(t), Q^{2*}_R(t))$ are two different positive $\tau-$periodic solutions of system (2.5). Then we have $(Q^{1*}_S(0), Q^{1*}_R(0)) \neq (Q^{2*}_S(0), Q^{2*}_R(0))$. Note that any two points in $\mathbb{R}^2$ are order related either in $\leq$ or $\leq_K$. Without loss of generality, we can assume that either $(Q^{2*}_S(0), Q^{2*}_R(0)) < (Q^{1*}_S(0), Q^{1*}_R(0))$, or $(Q^{2*}_S(0), Q^{2*}_R(0)) <_K (Q^{1*}_S(0), Q^{1*}_R(0))$.

In the case where $(Q^{2*}_S(0), Q^{2*}_R(0)) < (Q^{1*}_S(0), Q^{1*}_R(0))$, we have

$$(Q^{2*}_S(\tau), Q^{2*}_R(\tau)) = (Q^{2*}_S(0), Q^{2*}_R(0)) < (Q^{1*}_S(0), Q^{1*}_R(0)) = (Q^{1*}_S(\tau), Q^{1*}_R(\tau)).$$

By the backward comparison theorem (see, e.g., [2, Lemma 2.2] and [9, Lemma 7.4.1]), it follows that $(Q^{2*}_S(t), Q^{2*}_R(t)) \leq (Q^{2*}_S(t), Q^{2*}_R(t))$ for all $t \in [0,\tau]$.

This should be corrected into $(Q^{2*}_S(t), Q^{2*}_R(t)) \leq (Q^{1*}_S(t), Q^{1*}_R(t))$ **for all** $t \in [0,\tau]$**.**

Without loss of generality, we assume that $Q^{2*}_S(0) < (Q^{1*}_S(0)$ **and define** $g(t) := Q^{1*}_S(t) - Q^{2*}_S(t)$**. Otherwise, we have** $Q^{2*}_R(0) < Q^{1*}_R(0)$ **and then choose**

$g(t) := Q_R^{1*}(t) - Q_R^{2*}(t)$**.** Then it follows from the first equation of (2.5) that

$$g'(t) = \left[f_S(S^*(t), Q_S^{1*}(t)) - \min\{\mu_S(Q_S^{1*}(t)), \mu_R(Q_R^{1*}(t))\}Q_S^{1*}(t)\right]$$
$$- \left[f_S(S^*(t), Q_S^{2*}(t)) - \min\{\mu_S(Q_S^{2*}(t)), \mu_R(Q_R^{2*}(t))\}Q_S^{2*}(t)\right].$$

Using (H1) and (H2), we see that $g(t)$ satisfies $g'(t) \leq 0$ **for all** $t \in [0, \tau]$**, and** $g'(t) < 0$ **when** $t \in [0, \tau]$ **is sufficiently close to** 0 **or** τ**. This implies that** $g(\tau) < g(0)$, **which contradicts the fact that** $g(t)$ is a $\tau-$periodic function.

In the case where $(Q_S^{2*}(0), Q_R^{2*}(0)) <_K (Q_S^{1*}(0), Q_R^{1*}(0))$, by the forward comparison theorem (see, e.g., [9, Theorem B.4]), we have

$$(Q_S^{2*}(t), Q_R^{2*}(t)) \leq_K (Q_S^{1*}(t), Q_R^{1*}(t)), \ \forall \ t \in [0, \tau]. \tag{2.8}$$

We divide the first (second) equation of (2.5) by $Q_S(t)$ ($Q_R(t)$) and integrating the resulting equation in $t \in [0, \tau]$, we have

$$0 = \int_0^\tau \frac{1}{Q_S^{1*}(t)} d(Q_S^{1*}(t)) = \int_0^\tau \frac{f_S(S^*(t), Q_S^{1*}(t))}{Q_S^{1*}(t)} dt - \int_0^\tau \min\{\mu_S(Q_S^{1*}(t)), \mu_R(Q_R^{1*}(t))\} dt,$$

and

$$0 = \int_0^\tau \frac{1}{Q_R^{1*}(t)} d(Q_R^{1*}(t)) = \int_0^\tau \frac{f_R(R^*(t), Q_R^{1*}(t))}{Q_R^{1*}(t)} dt - \int_0^\tau \min\{\mu_S(Q_S^{1*}(t)), \mu_R(Q_R^{1*}(t))\} dt.$$

Therefore,

$$\int_0^\tau \frac{f_S(S^*(t), Q_S^{1*}(t))}{Q_S^{1*}(t)} dt = \int_0^\tau \frac{f_R(R^*(t), Q_R^{1*}(t))}{Q_R^{1*}(t)} dt. \tag{2.9}$$

Similarly, we can prove that

$$\int_0^\tau \frac{f_S(S^*(t), Q_S^{2*}(t))}{Q_S^{2*}(t)} dt = \int_0^\tau \frac{f_R(R^*(t), Q_R^{2*}(t))}{Q_R^{2*}(t)} dt. \tag{2.10}$$

From (H2), (2.8) and the facts $(Q_S^{2*}(0), Q_R^{2*}(0)) <_K (Q_S^{1*}(0), Q_R^{1*}(0))$, we have

$$\int_0^\tau \frac{f_S(S^*(t), Q_S^{1*}(t))}{Q_S^{1*}(t)} dt \leq \int_0^\tau \frac{f_S(S^*(t), Q_S^{2*}(t))}{Q_S^{2*}(t)} dt, \tag{2.11}$$

Here I corrected your earlier $>$ into $\leq$, and only the third inequality in (H2) is used.

and

$$\int_0^\tau \frac{f_R(R^*(t), Q_R^{1*}(t))}{Q_R^{1*}(t)}dt \geq \int_0^\tau \frac{f_R(R^*(t), Q_R^{2*}(t))}{Q_R^{2*}(t)}dt. \tag{2.12}$$

Here I corrected your earlier $<$ into $\geq$, and only the third inequality in (H2) is used.

Note that when $t \in [0, \tau]$ is sufficiently close to zero or τ, we have $(Q_S^{2*}(t), Q_R^{2*}(t)) <_K (Q_S^{1*}(t), Q_R^{1*}(t))$, and hence, at least one of two inequalities in (2.11) and (2.12) is strict. This, together with (2.9) and (2.10), leads to a contradiction.

□

Next, we show that every solution $(S(t), R(t), Q_S(t), Q_R(t), u(t))$ of system (2.1) exists for all $t \geq 0$. By the continuation theorem, it suffices to prove that the solution of system (2.1) is bounded on finite time intervals. To this end, we let $T_S(t) = S(t) + Q_S(t)u(t)$ and $T_R(t) = R(t) + Q_R(t)u(t)$. Then $T_S(t)$ and $T_R(t)$ satisfy the linear systems (2.3) and (2.4), respectively. Hence, $T_S(t)$ and $T_R(t)$ are bounded on finite time intervals. So are $S(t)$, $R(t)$, $Q_S(t)u(t)$ and $Q_R(t)u(t)$. Since $Q_S(t) \geq Q_{\min,S}$ and $Q_R(t) \geq Q_{\min,R}$, it follows that $u(t)$ is bounded on finite time intervals. It remains to show $Q_S(t)$ and $Q_R(t)$ are bounded on finite time intervals. Let

$$\tilde{V}(t) = \frac{1}{2}[(Q_S(t))^2 + (Q_R(t))^2].$$

Since $S(t)$ and $R(t)$ in system (2.1) are bounded on finite time intervals, we can use the same arguments as in (2.7) to show that $\tilde{V}(t)$ is bounded in finite time intervals. So are $Q_S(t)$ and $Q_R(t)$. Thus, every solution of system (2.1) exists globally.

Let

$$Z_S = S^*(t) - S - Q_S u \text{ and } Z_R = R^*(t) - R - Q_R u. \tag{2.13}$$

Then (2.1) becomes

$$\begin{cases} \frac{dQ_S}{dt} = f_S(S^*(t) - Q_S u - Z_S, Q_S) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_S, \\ \frac{dQ_R}{dt} = f_R(R^*(t) - Q_R u - Z_R, Q_R) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_R, \\ \frac{du}{dt} = [\min\{\mu_S(Q_S), \mu_R(Q_R)\} - D]u, \\ \frac{dZ_S}{dt} = -DZ_S, \\ \frac{dZ_R}{dt} = -DZ_R, \end{cases} \tag{2.14}$$

with initial values in the domain

$$\mathbb{X} = \{(Q_S, Q_R, u, Z_S, Z_R) \in \mathbb{R}^5_+ : Q_N \geq Q_{\min,N},\ Q_N u + Z_N \leq N^*(0),\ N = S, R\}. \tag{2.15}$$

Biologically, $S(t) = S^*(t) - Q_S u - Z_S$ and $R(t) = R^*(t) - Q_R u - Z_R$ should be nonnegative. Indeed, if there exists a t_0 such that $S^*(t_0) - Q_S(t_0)u(t_0) - Z_S(t_0) = 0$ then

$$\begin{aligned} S'(t_0) &= (S^*(t) - Q_S u - Z_S)'(t_0) \\ &= D(S^{(0)} - S^*(t) + Q_S u + Z_S)(t_0) = DS^{(0)}(t_0) \geq 0, \end{aligned}$$

which implies that $S(t) \geq 0$ for all $t \geq 0$. Similarly, we can show that $R(t) \geq 0$ for all $t \geq 0$.

From the equations for u, Q_S and Q_R, along with (H1) and (H2) imply that $u(t) \geq 0$, $Q_S(t) \geq Q_{\min,S}$ and $Q_R(t) \geq Q_{\min,R}$ for all $t \geq 0$. Obviously, $Z_S(t),\ Z_R(t) \to 0$ as $t \to \infty$. Therefore, solutions of (2.1) are ultimately bounded on $\mathbf{X}$.

By putting $Z_S = 0$ and $Z_R = 0$ in (2.14), we obtain the following periodic limiting system:

$$\begin{cases} \frac{dQ_S}{dt} = f_S(S^*(t) - Q_S u, Q_S) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_S, \\ \frac{dQ_R}{dt} = f_R(R^*(t) - Q_R u, Q_R) - \min\{\mu_S(Q_S), \mu_R(Q_R)\}Q_R, \\ \frac{du}{dt} = [\min\{\mu_S(Q_S), \mu_R(Q_R)\} - D]u, \end{cases} \tag{2.16}$$

with initial values in the domain

$$\mathbb{Y} = \{(Q_S, Q_R, u) \in \mathbb{R}^3_+ : Q_N \geq Q_{\min,N},\ Q_N u \leq N^*(0),\ N = S, R\}. \tag{2.17}$$

Suppose $P : \mathbb{Y} \to \mathbb{Y}$ is the Poincaré map associated with system (2.16), that is,

$$P(Q_S(0), Q_R(0), u(0)) = (Q_S(\tau), Q_R(\tau), u(\tau)),\ \forall\ (Q_S(0), Q_R(0), u(0)) \in \mathbb{Y},$$

where $(Q_S(t), Q_R(t), u(t))$ is the unique solution of system (2.16). It is obvious that

$$P^n(Q_S(0), Q_R(0), u(0)) = (Q_S(n\tau), Q_R(n\tau), u(n\tau)),\ \forall\ n \geq 0.$$

From Lemma 2.1, it is easy to see that $(Q_S^*(t), Q_R^*(t), 0)$ is the trivial τ-periodic solution of (2.16). We will not discuss the local stability of $(Q_S^*(t), Q_R^*(t), 0)$ since the reaction functions in (2.16) are not differentiable. However, we are able to study the global stability of $(Q_S^*(t), Q_R^*(t), 0)$ by appealing to theory of monotone dynamical system (see, e.g., [6]). In the following, we first convert (2.16) to a cooperative system. Let

$$U_S = Q_S u \text{ and } U_R = Q_R u. \tag{2.18}$$

Then (2.16) becomes

$$\begin{cases} \frac{dU_S}{dt} = -DU_S + f_S(S^*(t) - U_S, \frac{U_S}{u})u, \\ \frac{dU_R}{dt} = -DU_R + f_R(R^*(t) - U_R, \frac{U_R}{u})u, \\ \frac{du}{dt} = [\min\{\mu_S(\frac{U_S}{u}), \mu_R(\frac{U_R}{u})\} - D]u, \end{cases} \tag{2.19}$$

with initial values in the domain

$$\mathbf{Y} = \{(U_S, U_R, u) \in \mathbb{R}_+^3 : u > 0,\ Q_{\min,N} u \le U_N \le N^*(0)\ \forall\ N = S, R\}.$$

It is easy to see system (2.19) is cooperative (see, e.g., [6]). Let $\mathbf{P} : \mathbf{Y} \to \mathbf{Y}$ be the Poincaré map associated with system (2.19), that is,

$$\mathbf{P}(U_S(0), U_R(0), u(0)) = (U_S(\tau), U_R(\tau), u(\tau)),\ \forall\ \tilde{x} := (U_S(0), U_R(0), u(0)) \in \mathbf{Y},$$

where $(U_S(t), U_R(t), u(t))$ is the unique solution of system (2.19).

For convenience, we define the time-average of a τ-periodic function as follows:

$$\langle f(t) \rangle := \frac{1}{\tau} \int_0^\tau f(t)dt.$$

The following result is concerned with the extinction of species for system (2.16):

Lemma 2.2. *Assume that $(Q_S(t), Q_R(t), u(t))$ is the solution of system (2.16) and $(Q_S(0), Q_R(0), u(0)) \in \mathbb{Y}$. If $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D \rangle < 0$, then*

$$\lim_{t \to \infty} |(Q_S(t), Q_R(t), u(t)) - (Q_S^*(t), Q_R^*(t), 0)| = 0, \tag{2.20}$$

where $(Q_S^*(t), Q_R^*(t))$ is the unique periodic solution of system (2.5).

Proof. We first show that every solution of system (2.16) in $\mathbb{Y}$ satisfies

$$\lim_{t \to \infty} u(t) = 0. \tag{2.21}$$

In fact, if $u(0) = 0$, then $u(t) \equiv 0$, so this is obvious. Now, we consider the case where $u(0) > 0$. Since systems (2.16) and (2.19) are essentially equivalent by using the relations defined in (2.18), it suffices to show that every solution of system (2.19) in $\mathbf{Y}$ satisfies (2.21). In system (2.19), we first extend $f_N(N, Q_N)$, $N = S, R$, to the the domain $\mathbb{R} \times [Q_{\min,N}, \infty)$ by defining

$$F_N(N, Q_N) = \begin{cases} f_N(N, Q_N) \quad \text{for } N \geq 0,\ Q_N \geq Q_{\min,N}, \\ 0 \quad \text{for } N < 0,\ Q_N \geq Q_{\min,N}. \end{cases}$$

Now, we consider the extended system corresponding to system (2.19):

$$\begin{cases} \frac{dU_S}{dt} = -DU_S + F_S(S^*(t) - U_S, \frac{U_S}{u})u, \\ \frac{dU_R}{dt} = -DU_R + F_R(R^*(t) - U_R, \frac{U_R}{u})u, \\ \frac{du}{dt} = [\min\{\mu_S(\frac{U_S}{u}), \mu_R(\frac{U_R}{u})\} - D]u. \end{cases} \tag{2.22}$$

Note that if $(U_S(0), U_R(0), u(0)) \in \mathbf{Y}$ and $(U_S(t), U_R(t), u(t))$ is a solution of system (2.19), then $(U_S(t), U_R(t), u(t))$ satisfies

$$u(t) > 0,\ Q_{\min,N} u(t) \leq U_N(t) \leq N^*(t),\ \forall\ N = S, R,\ t \geq 0. \tag{2.23}$$

This implies that $(U_S(t), U_R(t), u(t))$ is also a solution of system (2.22).

For any given $\delta > 0$, we assume that $\bar{u}(t) = \bar{u}(t, \delta)$ is the unique solution of

$$\begin{cases} \frac{d\bar{u}}{dt} = [\min\{\mu_S(Q_S^*(t)), \mu_R(Q_S^*(t))\} - D]\bar{u}, \\ \bar{u}(0) = \delta, \end{cases} \tag{2.24}$$

and we let

$$\bar{U}_S(t) = Q_S^*(t)\bar{u}(t) \text{ and } \bar{U}_R(t) = Q_R^*(t)\bar{u}(t). \tag{2.25}$$

It is easy to see that

$$\frac{\bar{U}_N(t)}{\bar{u}(t)} = Q_N^*(t) \geq Q_{\min,N}, \text{ for all } N = S, R \text{ and } t \geq 0.$$

However, we can not guarantee that $S^*(t) - \bar{U}_S(t) \geq 0$ and $R^*(t) - \bar{U}_R(t) \geq 0$, for all $t \geq 0$. This is the reason why we consider the system (2.22) instead of (2.19).

From (2.5), (2.24) and (2.25), we are ready to show the following inequalities:

$$\begin{cases} \frac{d\bar{U}_S}{dt} \geq -D\bar{U}_S + F_S(S^*(t) - \bar{U}_S, \frac{\bar{U}_S}{\bar{u}})\bar{u}, \\ \frac{d\bar{U}_R}{dt} \geq -D\bar{U}_R + F_R(R^*(t) - \bar{U}_R, \frac{\bar{U}_R}{\bar{u}})\bar{u}, \\ \frac{d\bar{u}}{dt} = [\min\{\mu_S(\frac{\bar{U}_S}{\bar{u}}), \mu_R(\frac{\bar{U}_R}{\bar{u}})\} - D]\bar{u}. \end{cases} \tag{2.26}$$

The third equation in (2.26) is obvious. The establishments of the first two inequalities in (2.26) are similar, so we only demonstrate the first one. By direct computations, it follows that

$$\begin{aligned}
\frac{d\bar{U}_S}{dt} &= \frac{dQ_S^*(t)}{dt}\bar{u}(t) + \frac{d\bar{u}(t)}{dt}Q_S^*(t) \\
&= [f_S(S^*(t), Q_S^*(t)) - \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\}Q_S^*(t)]\,\bar{u}(t) \\
&+ [\min\{\mu_S(Q_S^*(t)), \mu_R(Q_S^*(t))\} - D]\,\bar{u}(t)Q_S^*(t) \\
&= -DQ_S^*(t)\bar{u}(t) + f_S(S^*(t), Q_S^*(t))\bar{u}(t) = -D\bar{U}_S + f_S(S^*(t), \frac{\bar{U}_S}{\bar{u}})\bar{u} \\
&\geq -D\bar{U}_S + F_S(S^*(t) - \bar{U}_S, \frac{\bar{U}_S}{\bar{u}})\bar{u}.
\end{aligned}$$

Thus, (2.26) is valid. Note that

$$\bar{U}_S(0) = Q_S^*(0)\bar{u}(0) = Q_S^*(0)\cdot\delta \text{ and } \bar{U}_R(0) = Q_R^*(0)\bar{u}(0) = Q_R^*(0)\cdot\delta,$$

where δ appears in (2.24). Since $(U_S(0), U_R(0), u(0)) \in \mathbf{Y}$, it follows that $(U_S(t), U_R(t), u(t))$ satisfies (2.23), and hence, we can choose a suitable $\delta > 0$ such that

$$(\bar{U}_S(0), \bar{U}_R(0), \bar{u}(0)) \geq (U_S(0)), U_R(0), u(0)). \tag{2.27}$$

Since (2.22) is a cooperative system, it follows from the comparison theorem together with (2.26) and (2.27) that

$$(\bar{U}_S(t), \bar{U}_R(t), \bar{u}(t)) \geq (U_S(t)), U_R(t), u(t)),\ \forall\ t \geq 0.$$

This implies that

$$\bar{u}(t) \geq u(t),\ \forall\ t \geq 0. \tag{2.28}$$

On the other hand, the assumption $\langle\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle < 0$ implies that

$$\lim_{t\to\infty} \bar{u}(t) = 0. \tag{2.29}$$

Thus, (2.21) follows from (2.28) and (2.29).

Next, we are going to show that (2.20) holds. Recall that $P : \mathbb{Y} \to \mathbb{Y}$ is the Poincaré map associated with system (2.16). For any given $x^0 = (Q_S(0), Q_R(0), u(0)) \in \mathbb{Y}_0$, let $\omega(x^0)$ be the omega limit set of x^0 for the period map P associated with (2.16), that is,

$$\omega = \omega(x^0) := \{x^* : \exists\{m_k\} \to \infty \text{ such that } \lim_{k\to\infty} P^{m_k}(x^0) = x^*\}.$$

From (2.21), it follows that there exists a set $\mathcal{C} \subset \mathbb{R}_+^2$ such that

$$\omega = \mathcal{C} \times \{0\}.$$

For any given $(Q_S^0, Q_R^0) \in \mathcal{C}$, we have $(Q_S^0, Q_R^0, 0) \in \omega \subset \mathbb{Y}$. By the definition of $\mathbb{Y}$, it follows that $(Q_S^0, Q_R^0) \in \Gamma$. Thus, $\mathcal{C} \subset \Gamma$, where Γ is defined in (2.6). From the property of the set Γ, it is easy to see that $\mathcal{C} \neq \{(0,0)\}$.

Assume that $\hat{P} : \Gamma \to \Gamma$ is the Poincaré map associated with system (2.5) and we define

$$W^s(Q_S^*(0), Q_R^*(0)) := \{(Q_S^0, Q_R^0) : \hat{P}^m(Q_S^0, Q_R^0) \to (Q_S^*(0), Q_R^*(0)) \text{ as } m \to \infty\}.$$

By [12, Lemma 1.2.1′], ω is a compact, invariant and internal chain transitive set for P. Moreover, if $(Q_S^0, Q_R^0) \in \mathbb{R}_+^2$ with $(Q_S^0, Q_R^0, 0) \in \omega$, there holds

$$P\mid_\omega (Q_S^0, Q_R^0, 0) = (\hat{P}(Q_S^0, Q_R^0), 0).$$

It then follows that $\mathcal{C}$ is a compact, invariant and internal chain transitive set for $\hat{P} : \Gamma \to \Gamma$. Since $\mathcal{C} \neq \{(0,0)\}$ and $\{(Q_S^*(0), Q_R^*(0))\}$ is globally attractive for $\hat{P}$ in Γ, we have $\mathcal{C} \cap W^s(Q_S^*(0), Q_R^*(0)) \neq \phi$. From [12, Theorem 1.2.1], it follows that $\mathcal{C} = \{(Q_S^*(0), Q_R^*(0))\}$, and hence, $\omega = \{(Q_S^*(0), Q_R^*(0), 0)\}$. We complete the proof of (2.20).

□

Next, we study the existence, uniqueness and the global stability of positive periodic solutions for the limiting system (2.16). Let

$$\mathbb{Y}_0 := \{(Q_S, Q_R, u) \in \mathbb{Y} : u > 0\},\ \partial\mathbb{Y}_0 := \mathbb{Y} \backslash \mathbb{Y}_0.$$

Lemma 2.3. *Assume that $(Q_S(t), Q_R(t), u(t))$ is the solution of system (2.16). If $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0$, then system (2.16) admits a globally attractive positive τ-periodic solution $(\tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t))$ in $\mathbb{Y}_0$, that is, for any $(Q_S(0), Q_R(0), u(0)) \in \mathbb{Y}_0$, we have*

$$\lim_{t\to\infty} |(Q_S(t), Q_R(t), u(t)) - (\tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t))| = 0. \tag{2.30}$$

Proof. Our strategy of the proof is as follows: we first prove that the system (2.16) is uniformly persistent, and hence, system (2.16) admits at least one positive periodic solution. Then the uniqueness and global attractivity of positive periodic solution can be proved by appealing to the theory of monotone dynamical system for the system (2.19). Note that systems (2.16) and (2.19) are essentially equivalent.

Claim 1. The system (2.16) is uniformly persistent with respect to $(\mathbb{Y}_0, \partial\mathbb{Y}_0)$ in the sense that there is an $\eta > 0$ such that for any $(Q_S(0), Q_R(0), u(0)) \in \mathbb{Y}_0$, the solution $(Q_S(t), Q_R(t), u(t))$ of (2.16) satisfies

$$\liminf_{t\to\infty} u(t) \geq \eta,$$

and system (2.16) admits at least one positive periodic solution.

We shall apply the theory of uniform persistence for discrete-time dynamical systems. Recall that $P : \mathbb{Y} \to \mathbb{Y}$ is the Poincaré map associated with system (2.16). It is easy to see that $P(\mathbb{Y}_0) \subset \mathbb{Y}_0$. Further, P is point dissipative and compact since solutions of system (2.16) are ultimately bounded. Let

$$E_0 := \{(Q_S^*(0), Q_R^*(0), 0)\}.$$

Then E_0 is a fixed point of P and is a compact and isolated invariant set for P in $\partial\mathbb{Y}_0$.

In the case where $u(0) = 0$, we have $u(t) = 0$ for any $t \geq 0$. Thus, (Q_S, Q_R) satisfies system (2.5) and it follows from Lemma 2.1 that

$$\lim_{t\to\infty} |(Q_S(t), Q_R(t)) - (Q_S^*(t), Q_R^*(t))| = 0.$$

Consequently, $P : \mathbb{Y} \to \mathbb{Y}$ has the property that

$$\omega(Q_S(0), Q_R(0), u(0)) = \{(Q_S^*(0), Q_R^*(0), 0)\},\ \forall\ (Q_S(0), Q_R(0), u(0)) \in \partial\mathbb{Y}_0,$$

where $\omega(x^0)$ is the omega-limit set of the orbit of P with initial values x^0. It is obvious that there is no cycle in $\partial\mathbb{Y}_0$ from E_0 to E_0.

Note that $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0$ and the third component of E_0 is identically zero. By the same arguments as in [12, Lemma 5.1.1], it follows that there exists a $\delta_0 > 0$ such that E_0 is a uniform weak repeller for $\mathbb{Y}_0$ in the sense that

$$\limsup_{n\to\infty} \|P^n(Q_S(0), Q_R(0), u(0)) - E_0\| \geq \delta_0,$$

for any $(Q_S(0), Q_R(0), u(0)) \in \mathbb{Y}_0$. Therefore, E_0 is isolated in $\mathbb{Y}$ and $W^s(E_0) \cap \mathbb{Y}_0 = \emptyset$, where $W^s(E_0)$ is the stable set of E_0 (see [12]).

Since $P : \mathbb{Y} \to \mathbb{Y}$ is point dissipative and compact, we conclude from [12, Theorem 1.1.3] that there exists a global attractor A for P in $\mathbb{Y}$. By [12, Theorem

1.3.1] on strong repellers, $P : \mathbb{Y} \to \mathbb{Y}$ is uniformly persistent with respect to $(\mathbb{Y}_0, \partial\mathbb{Y}_0)$. It follows from [12, Theorem 1.3.6] that there exists a global attractor A_0 for P in $\mathbb{Y}_0$ and P has a fixed point $(\tilde{Q}_S, \tilde{Q}_R, \tilde{u}) \in \mathbb{Y}_0$. Thus, there exists a positive periodic solution for (2.16) corresponding to the fixed point of the period map. By [12, Theorem 3.1.1], it follows that the periodic semiflow $T(t) : \mathbb{Y} \to \mathbb{Y}$, $t \geq 0$, associated with (2.16) is uniformly persistent with respect to $(\mathbb{Y}_0, \partial\mathbb{Y}_0)$.

Next, we will show that the positive periodic solution is unique and globally asymptotically stable. Recall that $\mathbf{P} : \mathbf{Y} \to \mathbf{Y}$ is the Poincaré map associated with system (2.19). It is easy to see that $\mathbf{P} : \mathbf{Y} \to \mathbf{Y}$ is strongly monotone, and strictly subhomogeneous in the sense that $\mathbf{P}(\theta\tilde{x}) > \theta\mathbf{P}(\tilde{x})$, $\forall\, \tilde{x} \in \mathbf{Y}$, $\theta \in (0, 1)$ (see [12, section 2.3]).

Why is system (2.19) **strongly monotone?**

Why is system (2.19) **strictly subhomogeneous?**

It then follows from [12, Theorem 2.3.2] that the following observation is valid:

Claim 2. If $\mathbf{P} : \mathbf{Y} \to \mathbf{Y}$ admits a nonempty compact invariant set $K \subset \mathrm{Int}(\mathbb{R}^3_+)$, then $\mathbf{P}$ has a fixed point $e \gg 0$ such that every nonempty compact invariant set of $\mathbf{P}$ in $\mathrm{Int}(\mathbb{R}^3_+)$ consists of e.

For any $\tilde{x} := (U_S(0), U_R(0), u(0)) \in \mathbf{Y}$, the solution $(U_S(t), U_R(t), u(t))$ of (2.19) satisfies $(\frac{U_S(0)}{u(0)}, \frac{U_R(0)}{u(0)}, u(0)) \in \mathbb{Y}_0$ and $(\frac{U_S(t)}{u(t)}, \frac{U_R(t)}{u(t)}, u(t))$ is a solution of system (2.16). From Claim 1, there is an $\eta > 0$ such that

$$\liminf_{t\to\infty} u(t) \geq \eta. \tag{2.31}$$

Then (2.18) and (2.31) imply that $\tilde{\omega}(\tilde{x}) \subset \mathrm{Int}(\mathbb{R}^3_+)$, where $\tilde{\omega}(\tilde{x})$ is the omega-limit set of the orbit through $\tilde{x} \in \mathbf{Y}$ for $\mathbf{P}$. By Claim 2, there is an $e \gg 0$ such that $\tilde{\omega}(\tilde{x}) = \{e\}$ for all $\tilde{x} \in \mathbf{Y}$. This implies that e is globally attractive for $\mathbf{P}$ in $\mathbf{Y}$. Corresponding to the fixed point of the period map $\mathbf{P}$, the system (2.19) has a globally attractive positive τ-periodic solution $(\tilde{U}_S(t), \tilde{U}_R(t), \tilde{u}(t))$ in $\mathbf{Y}$. Let $\tilde{Q}_S(t) = \frac{\tilde{U}_S(t)}{\tilde{u}(t)}$ and $\tilde{Q}_R(t) = \frac{\tilde{U}_R(t)}{\tilde{u}(t)}$. Then (2.16) has a globally attractive positive τ-periodic solution $(\tilde{Q}_S, \tilde{Q}_R, \tilde{u})$ in $\mathbb{Y}_0$. This ends the proofs. □

Let

$$\mathbf{X}_0 := \{(S, R, Q_S, Q_R, u) \in \mathbf{X} : u > 0\},\ \partial\mathbf{X}_0 := \mathbf{X}\backslash\mathbf{X}_0,$$

where $\mathbf{X}$ is defined in (2.2).

Theorem 2.1. *Let $(Q^*_S(t), Q^*_R(t))$ be the unique periodic solution of system* (2.5)*; $S^*(t)$ and $R^*(t)$ be the globally attractive positive τ-periodic solutions of the linear* *equations* (2.3) *and* (2.4)*, respectively. Assume that* $(S(t), R(t), Q_S(t), Q_R(t), u(t))$ *is the solution of system* (2.1)*. Then the following statements are valid:*

(i) *If* $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle < 0$*, then for any*

$$(S(0), R(0), Q_S(0), Q_R(0), u(0)) \in \mathbf{X},$$

we have

$$\lim_{t\to\infty} |(S(t), R(t), Q_S(t), Q_R(t), u(t)) - (S^*(t), R^*(t), Q_S^*(t), Q_R^*(t), 0)| = 0;$$

(ii) *If* $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0$*, then the system* (2.1) *admits a globally attractive positive* τ*-periodic solution* $(\tilde{S}(t), \tilde{R}(t), \tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t))$ *in* $\mathbf{X}_0$*, that is, for any* $(S(0), R(0), Q_S(0), Q_R(0), u(0)) \in \mathbf{X}_0$*, we have*

$$\lim_{t\to\infty} |(S(t), R(t), Q_S(t), Q_R(t), u(t)) - (\tilde{S}(t), \tilde{R}(t), \tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t))| = 0, \tag{2.32}$$

where $\tilde{S}(t) = S^*(t) - \tilde{Q}_S(t)\tilde{u}(t)$ *and* $\tilde{R}(t) = R^*(t) - \tilde{Q}_R(t)\tilde{u}(t)$*.*

Proof. Since system (2.1) and (2.14) are equivalent, it suffices to analyze the system (2.14). Let

$$\mathbb{X}_0 := \{(Q_S, Q_R, u, Z_S, Z_R) \in \mathbb{X} : u > 0\},\ \partial\mathbb{X}_0 := \mathbb{X}\setminus\mathbb{X}_0,$$

where $\mathbb{X}$ is defined in (2.15). Let $P_1 : \mathbb{X} \to \mathbb{X}$ be the Poincaré map associated with system (2.14) and $\omega_1(x^1)$ be the omega-limit set of the orbit of P_1 with initial values $x^1 \in \mathbb{X}$.

From the last two equations of (2.14), it follows that $\lim_{t\to\infty} Z_S(t) = 0$ and $\lim_{t\to\infty} Z_R(t) = 0$. Thus, there exists a set $\mathcal{I} \subset \mathbb{R}_+^3$ such that $\omega_1(x^1) = \mathcal{I} \times \{(0,0)\}$. For any given $(Q_S^0, Q_R^0, u^0) \in \mathcal{I}$, we have $(Q_S^0, Q_R^0, u^0, 0, 0) \in \omega_1(x^1) \subset \mathbb{X}$. By the definition of $\mathbb{X}$, it follows that $(Q_S^0, Q_R^0, u^0) \in \mathbb{Y}$. Thus, $\mathcal{I} \subset \mathbb{Y}$, where $\mathbb{Y}$ is defined in (2.17).

By [12, Lemma 1.2.1′], $\omega_1(x^1)$ is a compact, invariant and internal chain transitive set for P_1. Moreover, if $x^0 \in \mathbb{R}_+^3$ with $(x^0, 0, 0) \in \omega_1(x^1)$, there holds

$$P_1 |_{\omega_1(x^1)} (x^0, 0, 0) = (P(x^0), 0, 0),$$

where $P : \mathbb{Y} \to \mathbb{Y}$ is the Poincaré map associated with system (2.16). It then follows that $\mathcal{I}$ is a compact, invariant and internal chain transitive set for $P : \mathbb{Y} \to \mathbb{Y}$.

In the case where $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle < 0$, it follows from Lemma 2.2 that (2.16) has a globally attractive positive τ-periodic solution $(Q_S^*(t), Q_R^*(t), 0)$ in $\mathbb{Y}$. This implies that the unique fixed point $(Q_S^*(0), Q_R^*(0), 0)$ is an isolated invariant set in $\mathbb{Y}$ and no cycle connecting $(Q_S^*(0), Q_R^*(0), 0)$ to itself in $\mathbb{Y}$. Since $\mathcal{I}$ is a compact, invariant and internal chain transitive set for $P : \mathbb{Y} \to \mathbb{Y}$, it follows from a convergence theorem (see, e.g., [12, Theorem 1.2.1]) that $\mathcal{I}$ is a fixed point of P. That is, $\mathcal{I} = \{(Q_S^*(0), Q_R^*(0), 0)\}$, and hence, $\omega_1(x^1) = \mathcal{I} \times \{(0,0)\} = \{(Q_S^*(0), Q_R^*(0), 0, 0, 0)\}$. This implies that $(Q_S^*(0), Q_R^*(0), 0, 0, 0)\}$ is globally attractive for P_1 in $\mathbb{X}$. Corresponding to the fixed point of the period map P_1, system (2.14) has a globally attractive positive τ-periodic solution $(Q_S^*(t), Q_R^*(t), 0, 0, 0)\}$ in $\mathbb{X}$. In view of (2.13), we see that statement (i) holds true.

In the case where $\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0$, it follows from Lemma 2.3 that (2.16) has a globally attractive positive τ-periodic solution $(\tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t))$ in $\mathbb{Y}_0$. Note that $(Q_S^*(t), Q_R^*(t), 0)$ is also a τ-periodic solution of (2.16). This implies that the possible fixed points $(Q_S^*(0), Q_R^*(0), 0)$ and $(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0))$ are isolated invariant sets in $\mathbb{Y}$ and no subset of $\{(Q_S^*(0), Q_R^*(0), 0)\} \cup \{(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0))\}$ forms a cycle in $\mathbb{Y}$. Since $\mathcal{I}$ is a compact, invariant and internal chain transitive set for $P : \mathbb{Y} \to \mathbb{Y}$, it follows from a convergence theorem (see, e.g., [12, Theorem 1.2.2]) that either $\mathcal{I} = \{(Q_S^*(0), Q_R^*(0), 0)\}$ or $\mathcal{I} = \{(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0))\}$.

Suppose, by contradiction, that $\mathcal{I} = \{(Q_S^*(0), Q_R^*(0), 0)\}$. This implies that

$$\begin{aligned} P_1^n(Q_S(0), Q_R(0), u(0), Z_S(0), Z_R(0)) &:= (Q_S(n\tau), Q_R(n\tau), u(n\tau), Z_S(n\tau), Z_R(n\tau)) \\ &\to (Q_S^*(0), Q_R^*(0), 0, 0, 0) \text{ as } n \to \infty. \end{aligned}$$

Equivalently,

$$\lim_{t\to\infty} |(Q_S(t), Q_R(t), u(t), Z_S(t), Z_R(t)) - (Q_S^*(t), Q_R^*(t), 0, 0, 0)| = 0,$$

and hence, $\lim_{t\to\infty} |(Q_S(t), Q_R(t)) - (Q_S^*(t), Q_R^*(t))| = 0$. Let

$$\epsilon_0 := \frac{1}{2}\langle \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0.$$

Then there is a $T > 0$ such that

$$\min\{\mu_S(Q_S(t)), \mu_R(Q_R(t))\} - D \geq \min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D - \epsilon_0, \ \forall\, t \geq T.$$

By the third equation of (2.14), it follows that

$$\frac{du(t)}{dt} \geq [\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D - \epsilon_0]u, \ \forall\, t \geq T.$$

Without loss of generality, let $T = n_1\tau$ and $t = n_2\tau$ for some natural number n_1 and n_2 with $n_2 > n_1$, we obtain

$$u(n_2\tau) \geq u(n_1\tau)\exp\left[\int_{n_1\tau}^{n_2\tau}[\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D - \epsilon_0]dt\right].$$

Since $Q_S^*(t)$ and $Q_R^*(t)$ are τ-periodic functions, it follows that

$$u(n_2\tau) \geq u(n_1\tau)\exp\left[(n_2 - n_1)\tau(\langle\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle - \epsilon_0)\right],$$

that is,

$$u(n_2\tau) \geq u(n_1\tau)\exp\left[\frac{(n_2 - n_1)\tau}{2}\langle\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle\right].$$

Since $\langle\min\{\mu_S(Q_S^*(t)), \mu_R(Q_R^*(t))\} - D\rangle > 0$, it follows that

$$u(n_2\tau) \to \infty \text{ as } n_2 \to \infty,$$

which is a contradiction. Thus, we have $\mathcal{I} = \{(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0))\}$, and hence,

$$\omega_1(x^1) = \mathcal{I} \times \{(0,0)\} = \{(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0), 0, 0)\}.$$

This implies that $(\tilde{Q}_S(0), \tilde{Q}_R(0), \tilde{u}(0), 0, 0)$ is globally attractive for P_1 in $\mathbb{X}$. Corresponding to the fixed point of the period map P_1, system (2.14) has a globally attractive positive τ-periodic solution $(\tilde{Q}_S(t), \tilde{Q}_R(t), \tilde{u}(t), 0, 0)\}$ in $\mathbb{X}$. By virtue of (2.13), it follows that statement (ii) is valid. $\square$

3 Two Species Competition Model

In this section, we study the coexistence periodic solutions and uniform persistence for two species phytoplankton model (1.1). The following set is the region of interest for system (1.1):

$$\Omega = \{(S, R, Q_{S1}, Q_{R1}, u_1, Q_{S2}, Q_{R2}, u_2) \in \mathbb{R}_+^8 : Q_{Si} \geq Q_{\min,Si},\ Q_{Ri} \geq Q_{\min,Ri},\ i = 1, 2\}.$$

It is easy to show that Ω is positively invariant for (1.1) and any solution of (1.1) with initial value in Ω exists globally on $[0, \infty)$.

Why does every solution of (1.1) with initial value in Ω exist globally on $[0, \infty)$? It is better to give arguments supporting this claim.

Let

$$W_S(t) = S^*(t) - S - Q_{S1}u_1 - Q_{S2}u_2 \text{ and } W_R(t) = R^*(t) - R - Q_{R1}u_1 - Q_{R2}u_2, \quad (3.1)$$

where $S^*(t)$ and $R^*(t)$ are the globally attractive positive τ-periodic solutions of (2.3) and (2.4), respectively. Then W_S and W_R satisfy $\frac{dW_S}{dt} = -DW_S$ and $\frac{dW_R}{dt} = -DW_R$, respectively. Obviously,

$$\lim_{t\to\infty} W_S(t) = 0 \text{ and } \lim_{t\to\infty} W_R(t) = 0. \quad (3.2)$$

Therefore, solutions of (1.1) are ultimately bounded on Ω.

From Lemma 2.1, we assume that $(Q^*_{Si}(t), Q^*_{Ri}(t))$ is the globally attractive positive $\tau-$periodic solution for the system (2.5) with $f_S = f_{Si}$, $f_R = f_{Ri}$, $\mu_S = \mu_{Si}$ and $\mu_R = \mu_{Ri}$, i= 1, 2, respectively. This implies that

$$(S^*(t), R^*(t), Q^*_{S1}(t), Q^*_{R1}(t), 0, Q^*_{S2}(t), Q^*_{R2}(t), 0)$$

is the trivial $\tau-$periodic solution for (1.1).

Assume that

$$\langle \min\{\mu_{Si}(Q^*_{Si}(t)), \mu_{Ri}(Q^*_{Ri}(t))\} - D\rangle > 0, \ i = 1, 2. \quad (3.3)$$

Since $\langle \min\{\mu_{S1}(Q^*_{S1}(t)), \mu_{R1}(Q^*_{R1}(t))\} - D\rangle > 0$, it follows from Theorem 2.1 that we may assume $(\bar S(t), \bar R(t), \bar Q_{S1}(t), \bar Q_{R1}(t), \bar u_1(t))$ is the globally attractive positive τ-periodic solution of the system (2.1) in its feasible domain with $f_S = f_{S1}$, $f_R = f_{R1}$, $\mu_S = \mu_{S1}$ and $\mu_R = \mu_{R1}$, that is,

$$\begin{cases} \frac{dS}{dt} = (S^{(0)}(t) - S)D - f_{S1}(S, Q_{S1})u_1, \\ \frac{dR}{dt} = (R^{(0)}(t) - R)D - f_{R1}(R, Q_{R1})u_1, \\ \frac{dQ_{S1}}{dt} = f_{S1}(S, Q_{S1}) - \min\{\mu_{S1}(Q_{S1}), \mu_{R1}(Q_{R1})\}Q_{S1}, \\ \frac{dQ_{R1}}{dt} = f_{R1}(R, Q_{R1}) - \min\{\mu_{S1}(Q_{S1}), \mu_{R1}(Q_{R1})\}Q_{R1}, \\ \frac{du_1}{dt} = [\min\{\mu_{S1}(Q_{S1}), \mu_{R1}(Q_{R1})\} - D]u_1. \end{cases} \quad (3.4)$$

By the arguments similar to those in Lemma 2.1, we may assume $(\bar Q_{S2}(t), \bar Q_{R2}(t))$ is the globally attractive positive $\tau-$periodic solution for the following system:

$$\begin{cases} \frac{dQ_{S2}}{dt} = f_{S2}(\bar S(t), Q_{S2}) - \min\{\mu_{S2}(Q_{S2}), \mu_{R2}(Q_{R2})\}Q_{S2}, \\ \frac{dQ_{R2}}{dt} = f_{R2}(\bar R(t), Q_{R2}) - \min\{\mu_{S2}(Q_{S2}), \mu_{R2}(Q_{R2})\}Q_{R2}. \end{cases} \quad (3.5)$$

This implies that $(\bar{S}(t), \bar{R}(t), \bar{Q}_{S1}(t), \bar{Q}_{R1}(t), \bar{u}_1(t), \bar{Q}_{S2}(t), \bar{Q}_{R2}(t), 0)$ is the semi-trivial $\tau-$periodic solution for (1.1).

Similarly, we assume that $(\hat{S}(t), \hat{R}(t), \hat{Q}_{S1}(t), \hat{Q}_{R1}(t), 0, \hat{Q}_{S2}(t), \hat{Q}_{R2}(t), \hat{u}_2(t))$ is the semi-trivial $\tau-$periodic solution for (1.1), where $(\hat{S}(t), \hat{R}(t), \hat{Q}_{S2}(t), \hat{Q}_{R2}(t), \hat{u}_2(t))$ is the globally attractive positive τ-periodic solution of the system (2.1) in its feasible domain with $f_S = f_{S2}$, $f_R = f_{R2}$, $\mu_S = \mu_{S2}$ and $\mu_R = \mu_{R2}$ and $(\hat{Q}_{S1}(t), \hat{Q}_{R1}(t))$ is the globally attractive positive $\tau-$periodic solution for the following system:

$$\begin{cases} \frac{dQ_{S1}}{dt} = f_{S1}(\hat{S}(t), Q_{S1}) - \min\{\mu_{S1}(Q_{S1}), \mu_{R1}(Q_{R1})\}Q_{S1}, \\ \frac{dQ_{R1}}{dt} = f_{R1}(\hat{R}(t), Q_{R1}) - \min\{\mu_{S1}(Q_{S1}), \mu_{R1}(Q_{R1})\}Q_{R1}. \end{cases} \tag{3.6}$$

We further impose the following assumptions:

$$\langle \min\{\mu_{S1}(\hat{Q}_{S1}(t), \mu_{R1}(\hat{Q}_{R1}(t))\} - D\rangle > 0, \tag{3.7}$$

and

$$\langle \min\{\mu_{S2}(\bar{Q}_{S2}(t), \mu_{R2}(\bar{Q}_{R2}(t))\} - D\rangle > 0. \tag{3.8}$$

Next, we present the uniform persistence for the system (1.1). Let

$$\Omega_0 := \{(S, R, Q_{S1}, Q_{R1}, u_1, Q_{S2}, Q_{R2}, u_2) \in \Omega : u_1 > 0,\ u_2 > 0\},\ \partial\Omega_0 := \Omega\backslash\Omega_0.$$

Then

Theorem 3.1. *Let* (3.3), (3.7) *and* (3.8) *hold. Then system* (1.1) *is uniformly persistent with respect to* $(\Omega_0, \partial\Omega_0)$ *in the sense that there is an* $\eta > 0$ *such that for any* $(S(0), R(0), Q_{S1}(0), Q_{R1}(0), u_1(0), Q_{S2}(0), Q_{R2}(0), u_2(0)) \in \Omega_0$*, the solution* $(S(t), R(t), Q_{S1}(t), Q_{R1}(t), u_1(t), Q_{S2}(t), Q_{R2}(t), u_2(t))$ *of* (1.1) *satisfies*

$$\liminf_{t\to\infty} u_i(t) \geq \eta,\ i = 1, 2.$$

Further, system (1.1) *admits at least one positive periodic solution.*

Proof. Let $P_2 : \Omega \to \Omega$ be the Poincaré map associated with system (1.1), that is,

$$\begin{aligned} &P_2(S(0), R(0), Q_{S1}(0), Q_{R1}(0), u_1(0), Q_{S2}(0), Q_{R2}(0), u_2(0)) \\ &= ((S(\tau), R(\tau), Q_{S1}(\tau), Q_{R1}(\tau), u_1(\tau), Q_{S2}(\tau), Q_{R2}(\tau), u_2(\tau)), \end{aligned}$$

where $(S(0), R(0), Q_{S1}(0), Q_{R1}(0), u_1(0), Q_{S2}(0), Q_{R2}(0), u_2(0)) \in \Omega$. Let $\omega_2(x)$ be the omega-limit set of the orbit of P_2 through $x \in \Omega$. It is easy to see that

$P_2(\Omega_0) \subset \Omega_0$. Since solutions of the system (1.1) are ultimately bounded, it follows that P_2 is point dissipative and compact.

Let

$$M_0 = (S^*(0), R^*(0), Q^*_{S1}(0), Q^*_{R1}(0), 0, Q^*_{S2}(0), Q^*_{R2}(0), 0),$$

$$M_1 = (\bar{S}(0), \bar{R}(0), \bar{Q}_{S1}(0), \bar{Q}_{R1}(0), \bar{u}_1(0), \bar{Q}_{S2}(0), \bar{Q}_{R2}(0), 0),$$

and

$$M_2 = (\hat{S}(0), \hat{R}(0), \hat{Q}_{S1}(0), \hat{Q}_{R1}(0), 0, \hat{Q}_{S2}(0), \hat{Q}_{R2}(0), \hat{u}_2(0)).$$

Then M_0, M_1 and M_2 are fixed points of P_2 and are pairwise disjoint, compact and isolated invariant sets for P_2 in $\partial\Omega_0$. We are going to show the following property

$$\bigcup_{x \in \partial\Omega_0} \omega_2(x) \subset M_0 \cup M_1 \cup M_2. \tag{3.9}$$

In the case where $u_1(0) > 0$ and $u_2(0) = 0$, we have $u_1(t) > 0$ and $u_2(t) = 0$, $\forall\ t \geq 0$. Then $(S(t), R(t), Q_{S1}(t), Q_{R1}(t), u_1(t))$ satisfies the system (3.4). Since $\langle \mu_{S1}(Q^*_{S1}(t))\mu_{R1}(Q^*_{R1}(t)) - D\rangle > 0$, it follows from Theorem 2.1 that

$$\lim_{t\to\infty} |(S(t), R(t), Q_{S1}(t), Q_{R1}(t), u_1(t)) - (\bar{S}(t), \bar{R}(t), \bar{Q}_{S1}(t), \bar{Q}_{R1}(t), \bar{u}_1(t))| = 0.$$

Thus, the equations for $Q_{S2}(t)$ and $Q_{R2}(t)$ in (1.1) are asymptotic to the system (3.5). By the theory of asymptotically periodic semiflows (see, e.g., [11] or [12, section 3.2]) and the arguments similar to those in Lemma 2.1, it follows that

$$\lim_{t\to\infty} |(Q_{S2}(t), Q_{R2}(t)) - (\bar{Q}_{S2}(t), \bar{Q}_{R2}(t))| = 0.$$

Since

$$\begin{aligned} &P_2^n(S(0), R(0), Q_{S1}(0), Q_{R1}(0), u_1(0), Q_{S2}(0), Q_{R2}(0), 0) \\ &= ((S(n\tau), R(n\tau), Q_{S1}(n\tau), Q_{R1}(n\tau), u_1(n\tau), Q_{S2}(n\tau), Q_{R2}(n\tau), 0), \end{aligned}$$

it follows that

$$\lim_{n\to\infty} P_2^n(S(0), R(0), Q_{S1}(0), Q_{R1}(0), u_1(0), Q_{S2}(0), Q_{R2}(0), 0) = M_1.$$

In the case where $u_1(0) = 0$ and $u_2(0) > 0$, we can use the similar arguments to show that

$$\lim_{n\to\infty} P_2^n(S(0), R(0), Q_{S1}(0), Q_{R1}(0), 0, Q_{S2}(0), Q_{R2}(0), u_2(0)) = M_2.$$

In the case where $u_1(0) = 0$ and $u_2(0) = 0$, we can show that

$$\lim_{n\to\infty} P_2^n(S(0), R(0), Q_{S1}(0), Q_{R1}(0), 0, Q_{S2}(0), Q_{R2}(0), 0) = M_0.$$

Consequently, $P_2 : \Omega \to \Omega$ satisfies the property (3.9). It is obvious that no subset of $\{M_0, M_1, M_2\}$ forms a cycle in $\partial\Omega_0$.

Each M_j corresponds to a periodic solution with at least one component that is identically zero. By (3.3), (3.7), (3.8) and the same arguments as in [12, Lemma 5.1.1], there exists a $\delta > 0$ such that each M_j is a uniform weak repeller for Ω_0 in the sense that

$$\limsup_{n\to\infty} \|P_2^n(x) - M_j\| \geq \delta,$$

for any $x \in \Omega_0$. Therefore, each M_j is isolated in Ω and $\mathcal{W}^s(M_j) \cap \Omega_0 = \emptyset$, where $\mathcal{W}^s(M_j)$ is the stable set of M_j (see [12]).

Since $P_2 : \Omega \to \Omega$ is point dissipative and compact, we conclude from [12, Theorem 1.1.3] that there exists a global attractor $\mathcal{A}$ for P_2 in Ω. By [12, Theorem 1.3.1] on strong repellers, $P_2 : \Omega \to \Omega$ is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. It follows from [12, Theorem 1.3.6] that there exists a global attractor $\mathcal{A}_0$ for P_2 in Ω_0 and P_2 has at least one fixed point $x \in \Omega_0$. Thus, there exists a positive periodic solution for (1.1) corresponding to the fixed point x of the period map. By [12, Theorem 3.1.1], it follows that the periodic semiflow $\mathcal{T}(t) : \Omega \to \Omega,\ t \geq 0$, associated with (1.1) is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. This completes the proof. □

To finish this section, we remark that by a change of variables

$$U_{Si} = Q_{Si}u_i \text{ and } U_{Ri} = Q_{Ri}u_i,\ i = 1, 2, \tag{3.10}$$

together with (3.1), we can rewrite (1.1) as the following system:

$$\begin{cases} \frac{dU_{Si}}{dt} = -DU_{Si} + f_{Si}(S^*(t) - U_{S1} - U_{S2} - W_S, \frac{U_{Si}}{u_i})u_i, \\ \frac{dU_{Ri}}{dt} = -DU_{Ri} + f_{Ri}(R^*(t) - U_{R1} - U_{R2} - W_R, \frac{U_{Ri}}{u_i})u_i, \\ \frac{du_i}{dt} = [\min\{\mu_{Si}(\frac{U_{Si}}{u_i}), \mu_{Ri}(\frac{U_{Ri}}{u_i})\} - D]u_i,\ i = 1, 2, \\ \frac{dW_S}{dt} = -DW_S, \\ \frac{dW_R}{dt} = -DW_R, \end{cases} \tag{3.11}$$

with initial values in its feasible domain. In view of (3.2), we see that (3.11) has a limiting system:

$$\begin{cases} \frac{dU_{Si}}{dt} = -DU_{Si} + f_{Si}(S^*(t) - U_{S1} - U_{S2}, \frac{U_{Si}}{u_i})u_i, \\ \frac{dU_{Ri}}{dt} = -DU_{Ri} + f_{Ri}(R^*(t) - U_{R1} - U_{R2}, \frac{U_{Ri}}{u_i})u_i, \\ \frac{du_i}{dt} = [\min\{\mu_{Si}(\frac{U_{Si}}{u_i}), \mu_{Ri}(\frac{U_{Ri}}{u_i})\} - D]u_i, \ i = 1, 2. \end{cases} \tag{3.12}$$

It is easy to see that the Poincaré map associated with system (3.12) is monotone with respect to the partial order $\leq_K$ (see, e.g., [6]), which is induced by the positive cone $K := \mathbb{R}^3_+ \times \mathbb{R}^3_-$ in $\mathbb{R}^6$. Consequently, if system (1.1) admits at most one positive periodic solution, then we can conclude that the positive periodic solution obtained in Theorem 3.1 is globally attractive in Ω_0 by appealing to the theory of monotone systems. Note that the uniqueness of positive periodic solution of the high dimensional periodic system (1.1) remains a challenging problem, and we leave it for future investigation.

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