Chapter 1 Mathematical Modeling in Ecology

Ecology : The branch of biology that deals with the relationships between living organisms and their environment (Oxford English Dictionary)

" Ecology is the study of distribution and abundance of organisms "

Thus the population of a species is affected by space and time.

§1.1 Single Population Growth : Continuous model

(i) Malthusim Model

The economist Thomas Malthus construct the exponential model

$$\frac{dN}{dt} = rN \quad , N(0) = N_0 \quad . \qquad (1.1)$$

Assumption : Let $N(t)$ be the population density of species N at time t.
The rate of change of population density of a species N is proportional to the current population. r is called intrinsic growth

rate, the unit of r is $1/\text{time}$.

Then $N(t) = N_0 e^{rt}$

We may rewrite (1.1) as

$$\frac{1}{N}\frac{dN}{dt} = r = \text{per capita growth rate}$$

The model (1.1) is good for the organisms, like bacteria in lab., living in an enviroment with infinite supply of resource.

Density dependent population growth:

Due to the limited resource in the environment, Pierre-Francois Virhulst construct the famous Logistic equation

$$\frac{dN}{dt} = rN - \alpha N^2 \qquad (1.2)$$

where α is called intraspecific competition coefficients.

(1.2) can be rewritten as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \qquad (1.3)$$
$$N(0) = N_0$$

K is called carrying capacity of the environment.

~~Although~~ We can solve ~~the~~ logistic equation (1.3) by separation of variable

$$\frac{dN}{N(1-\frac{N}{K})} = r\,dt \qquad (1.4)$$

$$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{(1-N/K)} = \frac{1}{N} + \frac{(1/K)}{1-N/K}$$

$$1 = A(1-\frac{N}{K}) + BN$$

$$1 = A \qquad B = \frac{A}{K} = \frac{1}{K}$$

Integrating (1.4), we obtain

$$\ln N - \ln(1-\frac{N}{K}) = rt + C_1$$

$$\ln \frac{N}{1-\frac{N}{K}} = rt + C_1$$

$$\frac{N}{1-\frac{N}{K}} = Ce^{rt}, \quad C = e^{C_1}$$

At $t=0$, $\dfrac{N_0}{1-\frac{N_0}{K}} = C$

$$N = Ce^{rt}\left(1 - \frac{N}{K}\right)$$

$$N\left(1 + \frac{C}{K}e^{rt}\right) = Ce^{rt}$$

$$N(t) = \frac{Ce^{rt}}{1 + \frac{C}{K}e^{rt}} = \frac{N_0 e^{rt}}{\left(1 - \frac{N_0}{K}\right) + \frac{N_0}{K}e^{rt}}$$

Check! : $N(0) = N_0$, $\lim_{t\to\infty} N(t) = K$

~~The above formula will be used in curve fitting.~~

It is important to understand the qualitative behavior of the solution $N(t)$.

If $N(0) = N_0 < K$ then $N'(t) > 0$ for all t and $N(t) \to K$ as $t \to \infty$. Since

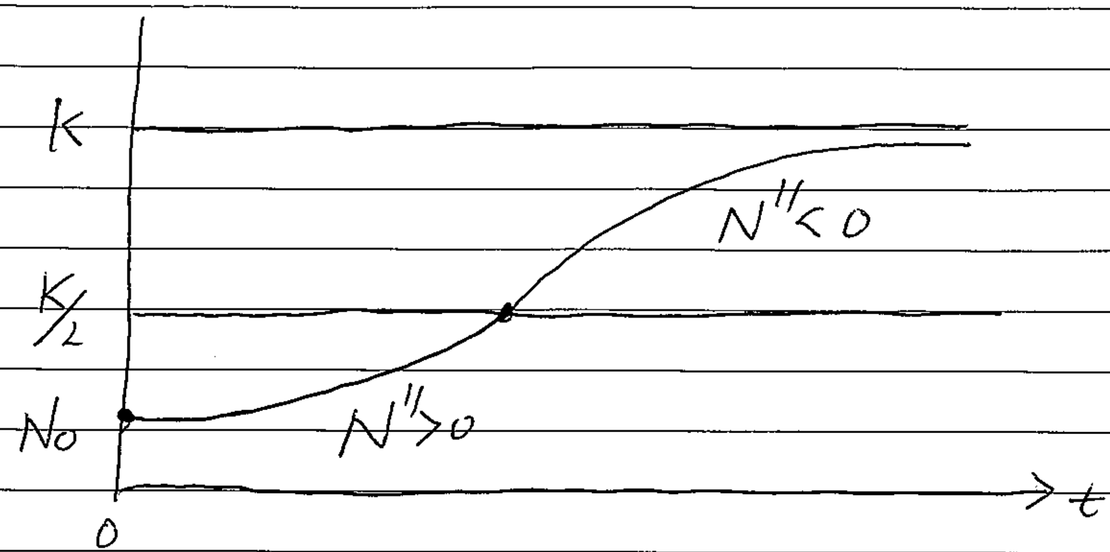
$$N''(t) = rN'(t)\left[1 - \frac{2N(t)}{K}\right]$$

it follows that

$$N''(t) > 0 \quad \text{if} \quad N(t) < K/2$$

$$N''(t) < 0 \quad \text{if} \quad N(t) > K/2$$

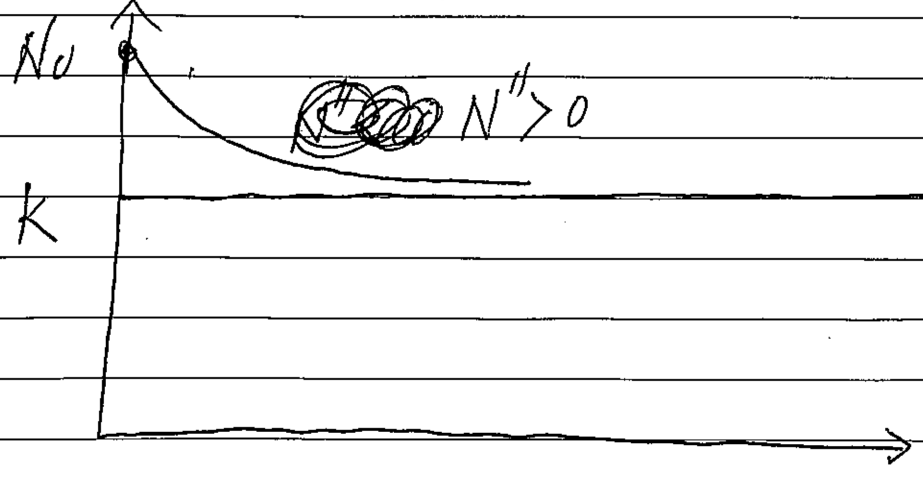
The behavior of the solution $N(t)$ is as follows:



The inflection point occurs at t^* where $N(t^*) = K/2$, half of the carrying capacity.

If $N(0) = N_0 > K$ then $N'(t) < 0$ for all $t > 0$

$N(t) > K$, $\lim_{t \to \infty} N(t) = K$, $N''(t) > 0$



Alternative to the logistic model:

(1) θ-logistic model Ayala 1973

$$N' = \frac{dN}{dt} = rN\left(1-\left(\frac{N}{K}\right)^{\theta}\right)$$

Sometimes from the experimental data, the inflection pt may not occurs at $K/2$.

$$N'' = rN'\left(1-2\left(\frac{N}{K}\right)^{\theta}\right)$$

$$N''>0 \iff 1/2 > \left(\frac{N}{K}\right)^{\theta} \iff \left(\frac{1}{2}\right)^{1/\theta} K > N$$

(2) Gompertz Growth (Applied to growth of solid tumor)

$$N' = \frac{dN}{dt} = rN \ln\left(\frac{K}{N}\right) = rN\left[\ln K - \ln N\right]$$

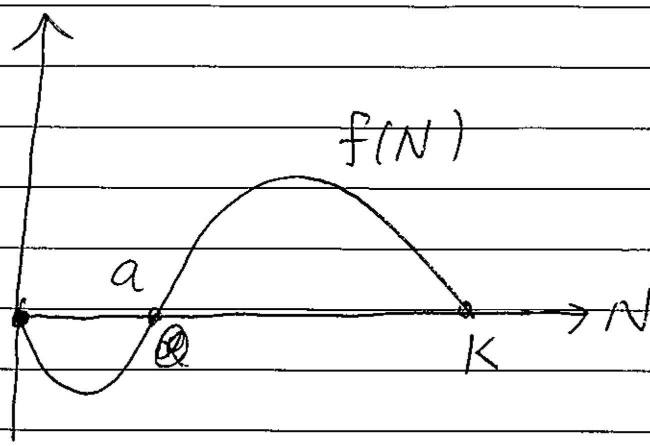
$$N'' = r\left[N\cdot\left(\frac{-1}{N}\right) + \ln\left(\frac{K}{N}\right)\right]$$

$$N''>0 \iff \ln\left(\frac{K}{N}\right) > 1$$

$$\iff K/N > e \iff K/e > N$$

(3) Allell's effect

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)(N-a) = f(N)$$



When the population density is too small,
~~Survive~~ the population becomes extinct.

$$0 < N(0) < a \Longrightarrow \lim_{t\to\infty} N(t) = 0$$

$$N(a) > a \Longrightarrow \lim_{t\to\infty} N(t) = K$$

Analysis:

One-dimensional flow

$$\frac{dN}{dt} = f(N)$$

Definition: We say that N^* is an equilibrium if

$$f(N^*) = 0$$

Local Stability of equilibrium :

Taylor expansion about $N=N^*$:

$$f(N) = f(N^*) + (N-N^*)\, f'(N^*) + \text{higher order term}$$
$$= (N-N^*)\, f'(N^*) + \text{higher order term}$$

Let $x = N - N^*$. Then

$$\frac{dx}{dt} = f'(N^*)\, x + \text{higher order term}$$

Linearized equation :

$$\begin{cases} \dfrac{dx}{dt} = f'(N^*)\, x \\ x(0) = N(0) - N^* \end{cases}$$

$$x(t) = x(0)\, e^{f'(N^*)t}$$

If $f'(N^*) < 0$ then $x(t) \to 0$ as $t \to \infty$

If $f'(N^*) > 0$ then $x(t) \to \infty$ as $t \to \infty$

Definition: We say that the equilibrium N^* of the differential equation $\frac{dN}{dt} = f(N)$, is locally stable if $f'(N^*) < 0$

In this case, if the initial condition $N(0)$ is close to N^* then it can be shown that

$$\lim_{t \to \infty} N(t) = N^*$$

If $f'(N^*) > 0$ then N^* is unstable, i.e. $N(t)$ will escape any neighborhood of N^*.

Stable: $\longrightarrow \underset{N^*}{\bullet} \longleftarrow$

Unstable: $\longleftarrow \underset{N^*}{\bullet} \longrightarrow$

Example: Logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = f(N)$$

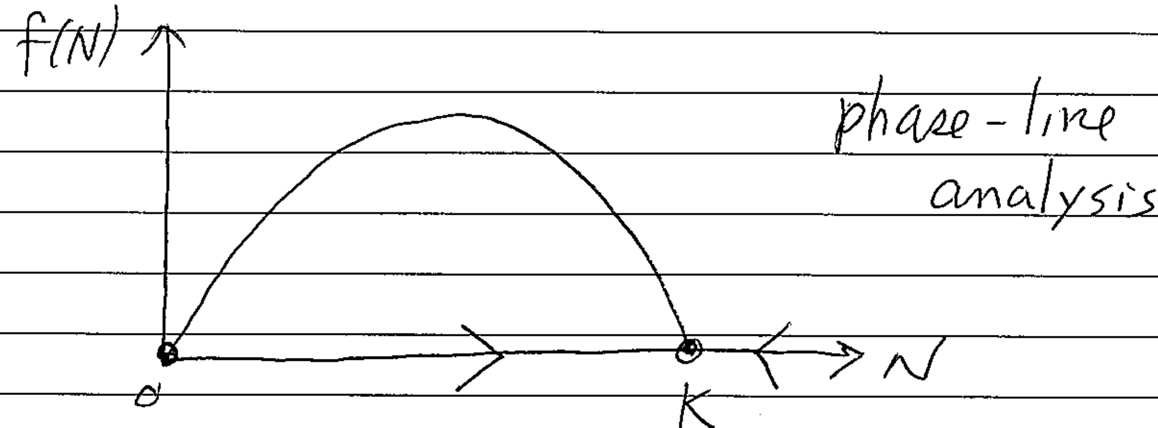
Per-Duet

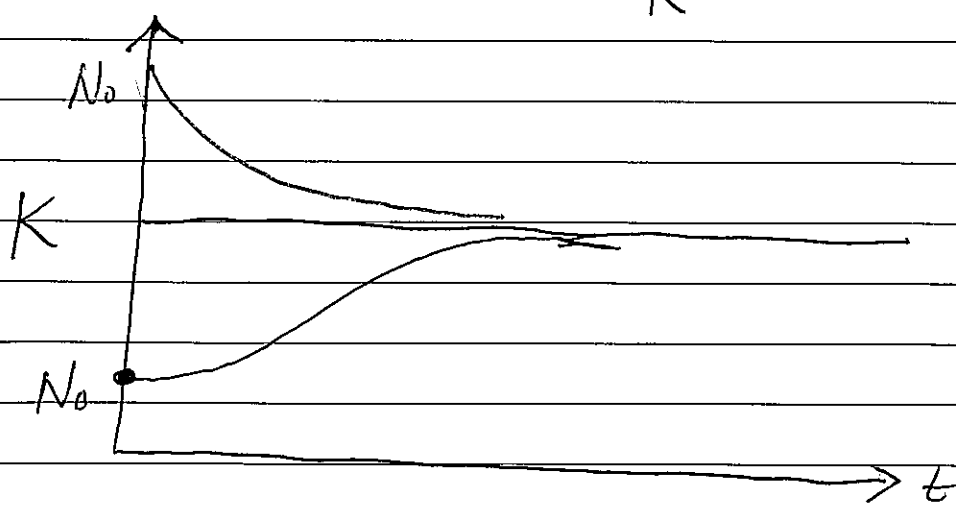
There are two equilibria $N^* = 0$, $N^* = K$

$$f'(N) = r\left(1 - \frac{2N}{K}\right)$$

$$f'(0) = r > 0, \quad f'(K) = -\frac{r}{K} < 0$$

Hence $N^* = 0$ is unstable, $N^* = K$ is stable





Example: Allell's effect

$$\frac{dN}{dt} = f(N) = rN\left(1-\frac{N}{K}\right)(N-a), \quad 0<a<K$$

$$f'(N) = rN\left(1-\frac{N}{K}\right) + (N-a)r\left[1-\frac{2N}{K}\right]$$

Three equilibria, $N=0$, $N=a$, $N=K$

$$f'(0) = -ra < 0$$

$$f'(a) = ra\left(1-\frac{a}{K}\right) > 0$$

$$f'(K) = (K-a)\,r\,(-1) < 0$$