ON THE ASYMPTOTIC BEHAVIOR OF
SOLUTIONS OF $v''(x) + x \sin v(x) = 0$

BY

WEI-MING NI* (倪維明) AND SZE-BI HSU** (許世璧)

Abstract. In this paper we study the asymptotic behavior of the
solution $v(x)$ of initial value problem (1.1) which arises from a
mathematical model describing the large deformations of a heavy
cantilever by its own weight.

1. Introduction. In this paper we are concerned with the
asymptotic behavior of the solutions of the following initial value
problem:

$$
  v''(x) + x \sin v(x) = 0,
  
  v'(0) = 0,
  
  v(0) = a, \quad 0 < a < \pi.
  
$$

(1.1)

The qualitative behavior of the solutions $v(x, a)$ of (1.1) is
important to the studies of the following mathematical model (1.2)
which describes the large deformations of a heavy cantilever by its
own weight (See [2] or [3]):

$$
  v''(x) + x \sin v(x) = 0,
  
  v'(0) = 0, \quad v(1) = \pi - \alpha, \quad 0 \leq \alpha \leq \pi.
  
$$

(1.2)

In [2] the authors studied the two-point boundary value problem
(1.2) by using shooting method. From the uniqueness of the
solutions of the initial value problem (1.1), it follows that

$$
  v(x, a) \equiv v(x, a + 2\pi),
  
  v(x, -a) \equiv -v(x, a),
  
  v(x, 0) \equiv 0, \quad v(x, \pi) \equiv \pi.
  
$$

Received by the editors December 18, 1987.

*) Research supported in part by NSF Grant DMS 8601246

**) Research supported in part by National Research Council, Republic of China.
Hence we restrict ourselves to study the case $0 < a < \pi$.

First we introduce the following Liapunov function

\[(1.3) \quad V(x) = (1 - \cos v(x)) + \frac{1}{2} \left( \frac{v'(x)}{x} \right)^2, \]

where $v(x) \equiv v(x, a)$.

It is easy to verify that

\[(1.4) \quad V'(x) = -\frac{1}{2} \left( \frac{v(x)}{x} \right)^2 \leq 0. \]

Then we have

\[1 - \cos v(x) \leq V(x) \leq V(0) = 1 - \cos a, \]

and it follows that $|v(x)| \leq a$ for all $x \geq 0$. We rewrite the equation in (1.1) as

\[(1.5) \quad v''(x) + x \left( \frac{\sin v(x)}{v(x)} \right) v(x) = 0. \]

Let $0 < \delta < \min_{0 \leq v \leq a} (\sin v/v)$. Using Sturm's comparison theorem [1], we compare (1.5) with

\[(1.6) \quad v''(x) + \delta v(x) = 0 \]

which is oscillatory over $[0, \infty)$. Thus the solution $v(x, a)$ is oscillatory over $[0, \infty)$ for $0 < a < \pi$. Moreover, from (1.3) and (1.4) the solution $v(x, a)$ is oscillatory with the decreasing amplitudes. In the next section we shall prove that

\[(1.7) \quad \lim_{x \to \infty} v(x, a) = 0 \quad \text{for} \quad 0 < a < \pi. \]

Consequently, if we denote the zero of $v$ by $x_1 < x_2 < \cdots < x_l < \cdots$, then we have $|x_l - x_{l-1}| \to 0$ as $l \to \infty$; or, more precisely,

\[(1.8) \quad x_{l+1}^{3/2} - x_l^{3/2} \to \frac{3}{2} \pi \quad \text{as} \quad l \to \infty. \]

2. **Main results.** The purpose of this section is to establish (1.7). First, we make the following change of variables:

\[(2.1) \quad y = x^{3/2}, \quad u(y, a) = v(x, a). \]

Then we have
SOLUTIONS OF $v''(x) + x \sin v(x) = 0$

$$v_x = \frac{3}{2} x^{1/2} u_x,$$

$$v_{xx} = \frac{9}{4} x u_{xx} + \frac{3}{4} x^{-1/2} u_x,$$

$$v_{xx} + x \sin v = \frac{9}{4} x \left[ u_{xx} + \frac{1}{3y} u_x + \frac{4}{9} \sin u \right].$$

Thus (1.1) becomes

$$(2.2) \quad u_{xx} + \frac{1}{3y} u_x + \frac{4}{9} \sin u = 0,$$

$$(2.3) \quad u(0) = a, \quad 0 < a < \pi,$$

$$(2.4) \quad u_x(0) = 0.$$  

We note that (2.4) follows directly from L'Hospital rule. Let $0 < c < d$ be any two real numbers. Multiplying $u_x$ on both sides of (2.2) and integrating the resulting identity from $c$ to $d$ yields

$$(2.5) \quad \frac{1}{2} (u_x(d))^2 - \frac{1}{2} (u_x(c))^2 + \int_c^d \frac{1}{3y} (u_x)^2 \, dy$$

$$+ \frac{4}{9} (\cos u(c) - \cos u(d)) = 0.$$

Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{2k} < \cdots$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{2k+1} < \cdots$ be the local maxima and local minima of $u(y)$ respectively. Since $v(x, a)$ is oscillatory over $[0, \infty)$ with decreasing amplitudes, from (2.1) so is $u(y, a)$. Assume

$$(2.6) \quad \lim_{k \to \infty} u(\tau_{2k}) = \xi \geq 0,$$

and

$$(2.7) \quad \lim_{k \to \infty} u(\gamma_{2k+1}) = \eta \leq 0.$$  

Now we state the following lemma and we defer the proof to the end of this section.

**Lemma 2.1.** There exists $C = C(a) > 0$ such that

$$(2.8) \quad |\tau_k - \gamma_{2k+1}| \leq C \quad \text{for all} \quad k \geq 1.$$
Assume that (2.8) hold. From (2.6), (2.7), (2.8) and Cauchy-Schwarz inequality it follows that for each $k \geq 1$

$$
\xi - \eta \leq |u(r_{2k}) - u(r_{2k-1})|
= \left| \int_{r_{2k-1}}^{r_{2k}} u_s(y) \, dy \right|
\leq \int_{r_{2k-1}}^{r_{2k}} |u_s(y)| \, dy
\leq (r_{2k} - r_{2k-1})^{1/2} \left[ \int_{r_{2k-1}}^{r_{2k}} (u_s(y))^2 \, dy \right]^{1/2}
\leq C^{1/2} \left[ \int_{r_{2k-1}}^{r_{2k}} (u_s(y))^2 \, dy \right]^{1/2}
$$
or

$$
\frac{\xi - \eta}{C^{1/2}} \leq \left[ \int_{r_{2k-1}}^{r_{2k}} (u_s(y))^2 \, dy \right]^{1/2}.
$$

Letting $c = 0$, $d = r_k$ in (2.5) and $k \to \infty$, we have that

$$
(2.10) \quad \int_0^\infty \frac{1}{y} (u_s(y))^2 \, dy < \infty.
$$

Considering the following inequality

$$
(2.11) \quad \int_{r_{k-1}}^{r_k} \frac{(u_s(y))^2}{y} \, dy \geq \frac{1}{r_k} \int_{r_{k-1}}^{r_k} (u_s(y))^2 \, dy,
$$

we see that (2.9) and (2.11) imply that

$$
(2.12) \quad \int_{r_{k-1}}^{r_k} \frac{(u_s(y))^2}{y} \, dy \geq \frac{1}{r_k} \frac{(\xi - \eta)^2}{C}.
$$

From Lemma 2.1 and $\lim_{k \to \infty} r_k = +\infty$, there exists $k_0 > 0$ such that

$$
(2.13) \quad \frac{1}{r_k} \geq \frac{1}{2r_{k-1}}, \quad \text{for every} \quad k \geq k_0.
$$

From (2.12), (2.13), (2.8), we have that for $k \geq k_0$

$$
\int_{r_{k-1}}^{r_k} \frac{(u_s(y))^2}{y} \, dy \geq \frac{1}{2r_{k-1}} \frac{(\xi - \eta)^2}{C}
= \frac{1}{2} \left( \frac{\xi - \eta}{C} \right) \cdot C \frac{1}{r_{k-1}}
\geq \frac{1}{2} \left( \frac{\xi - \eta}{C} \right)^2 \int_{r_{k-1}}^{r_k} \frac{1}{y} \, dy.
$$
Summing up (2.14) over $k \geq k_0$ yields

$$
(2.15) \quad \int_{r_{k+1}}^{\infty} \frac{(u_0(y))^2}{y} \, dy \geq \frac{1}{2} \left( \frac{\xi - \eta}{C} \right)^2 \int_{r_{k+1}}^{\infty} \frac{1}{y} \, dy.
$$

Therefore $\xi - \eta = 0$ since otherwise (2.15) and (2.10) would lead to a contradiction. Since $\xi \geq 0$ and $\eta \leq 0$, we have that $\xi = \eta = 0$ or $\lim_{y \to \infty} u(y) = 0$. Thus we complete the proof of (1.7).

**Proof of Lemma 2.1:**

Let $w(y) = y^{1/6} u(y)$. Then (2.2) becomes

$$
(2.16) \quad w_{yy} + \left( \frac{5}{36} y^2 + \frac{4}{9} \frac{\sin u(y)}{u(y)} \right) w = 0.
$$

Since

$$
\delta_0 \leq \frac{\sin u(y)}{u(y)} \leq 1 \quad \text{for all} \quad y \geq 0
$$

where

$$
\delta_0 = \frac{\sin a}{a},
$$

we compare (2.16) with

$$
(2.17) \quad w_{yy} + \left( \frac{4}{9} \delta_0 \right) w = 0.
$$

Let $z_1 < z_2 < \cdots < z_i < \cdots$ be the zeros of $u(y)$. Then from Sturm's comparison theorem it follows that

$$
|z_i - z_{i-1}| \leq \frac{\pi}{\sqrt{4 \delta_0 / 9}} = C / 2
$$

or

$$
|r_k - r_{k-1}| \leq C \quad \text{for all} \quad k \geq 1,
$$

and Lemma 2.1 is proved.

Since $w$ and $u$ have exactly the same zeros in $(0, \infty)$, it follows from (2.16), (1.7) and Sturm's comparison theorem that $|z_i - z_{i-1}| \to (3/2) \pi$ as $l \to \infty$. Thus (1.8) holds.
REFERENCES


School of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.

Institute of Applied Mathematics
National Tsing Hua University
Hsinchu, Taiwan, R.O.C.