§3.11 Predator-Prey models

Let $x(t)$ be the population density of prey, $y(t)$ be the population density of predator at time t. The general model for predator-prey interaction is following

$$\frac{dx}{dt} = xf(x,y)$$

$$\frac{dy}{dt} = yg(x,y) \qquad (11.1)$$

$$x(0) > 0, \quad y(0) > 0$$

Where $f(x,y)$ and $g(x,y)$ satisfy

$$\frac{\partial f}{\partial y} \le 0, \quad \frac{\partial g}{\partial x} \ge 0$$

In 1923 Volterra proposed a simple model to explain the oscillatory levels of a certain fish catches in Adriatic. The model takes the form

$$\frac{dx}{dt} = ax - bxy,$$

$$\frac{dy}{dt} = cxy - dy, \qquad (11.2)$$

$$x(0) > 0, \quad y(0) > 0.$$

In (11.2) we assume the prey grows exponentially in the absence of predation. The prey is consumed by predator with the amount bxy per unit time and is converted into the new population of predator at the rate cxy. d is the death rate of predator. We note that (11.2) was also derived by chemist Lotka in 1920 for the auto catalysis of chemical reaction $A \to B$. (11.2) is called Lotka-Volterra predator-prey model. In (11.2) we have following equilibria: $(0,0)$, (x^*, y^*) where $x^* = \frac{d}{c}$, $y^* = \frac{a}{b}$.

Consider the Jacobian matrix of (11.2) at (x,y).

$$J(E) = J(x,y)\begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}$$

If $E = (0,0)$ then

$$J(0,0)=\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$$

and $(0,0)$ is a saddle point with stable manifold y–axis and unstable manifold x–axis.

If $E=(x^*,y^*)$ then

$$J(x^*,y^*)=\begin{bmatrix} 0 & -bx^* \\ cy^* & 0 \end{bmatrix}$$

and (x^*,y^*) is a center. The linearization provides no information for nonlinear system (11.2).

Write (11.2) as

$$\frac{dx}{dt}=bx(y^*-y)$$

$$\frac{dy}{dt}=cy(x-x^*)$$

Elimination variable t, we obtain equation

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{cy(x-x^*)}{-bx(y-y^*)}, \tag{11.3}$$

In $x\ y$ phase plane. From (11.3) we have

$$\frac{x-x^*}{x}dx+\frac{b}{c}\frac{y-y^*}{y}dy=0, \tag{11.4}$$

Integrate (11.4) we obtain

$$V(x,y)=\int_{x^*}^{x}\frac{\xi-x^*}{\xi}d\xi+\frac{b}{c}\int_{y^*}^{y}\frac{\eta-y^*}{\eta}d\eta\equiv const.$$

or

$$V(x,y)=(x-x^*-x^*\ln\frac{x}{x^*})+\frac{b}{c}(y-y^*-y^*\ln\frac{y}{y^*})$$

$$\equiv H$$

Then each solution of (11.2) is a periodic solution and we obtain a series of “neutral” stable closed curves in $x\ y$ plane (See Fig 11.1).

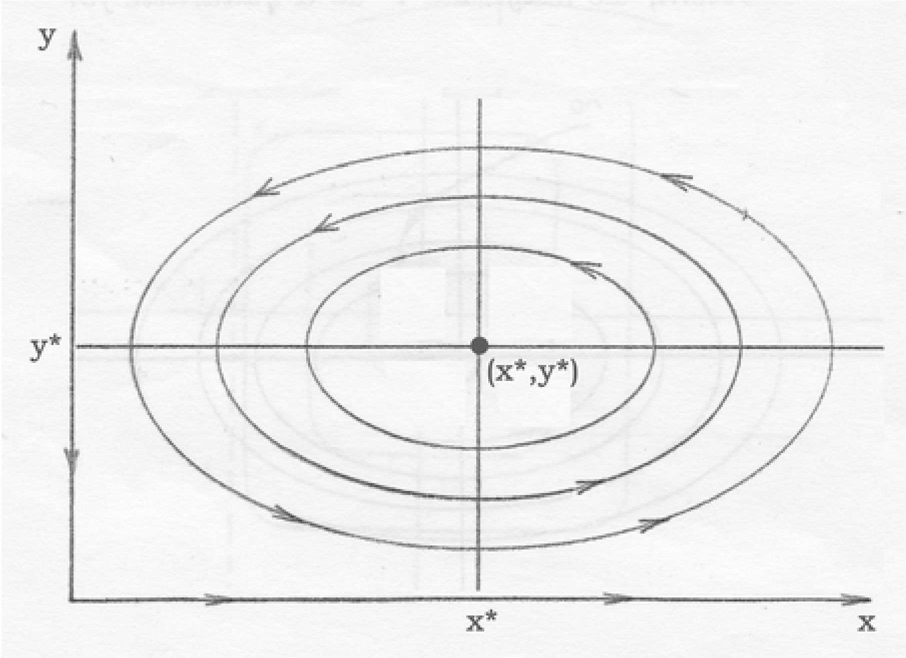

Figure 11.1

If we assume the prey grows logistically with carrying capacity K in the absence of predation, then the predator-prey model takes the form

$$\frac{dx}{dt} = rx(1-\frac{x}{K}) - bxy\,,$$

$$\frac{dy}{dt} = cxy - dy\,, \tag{11.5}$$

$$x(0) > 0\,,\quad y(0) > 0\,.$$

Then there are two cases

Case 1: $\beta = \frac{d}{c} > K$

Then there are two euqilibria: $(0,0)$ which is a saddle, $(K,0)$ which is a stable. From the isocline analysis we predict $(K,0)$ is global stable, i.e., every solution of (11.5) approach $(K,0)$ as $t \to \infty$ (See Fig. 11.2).

Case 2: $\beta = \frac{d}{c} < K$

Then there are three equilibria: $(0,0)$ which is a saddle, $(K,0)$ which is a saddle and (x^*, y^*), $x^* = \beta = \frac{d}{c}$, $y^* = \frac{r}{b}(1-\frac{x^*}{K})$ which is stable. From the isocline analysis we predict that (x^*, y^*) is global stable.

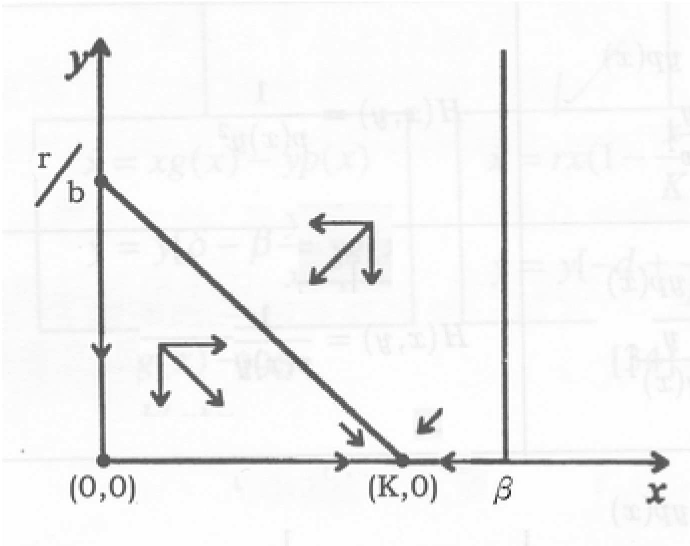

Figure 11.2

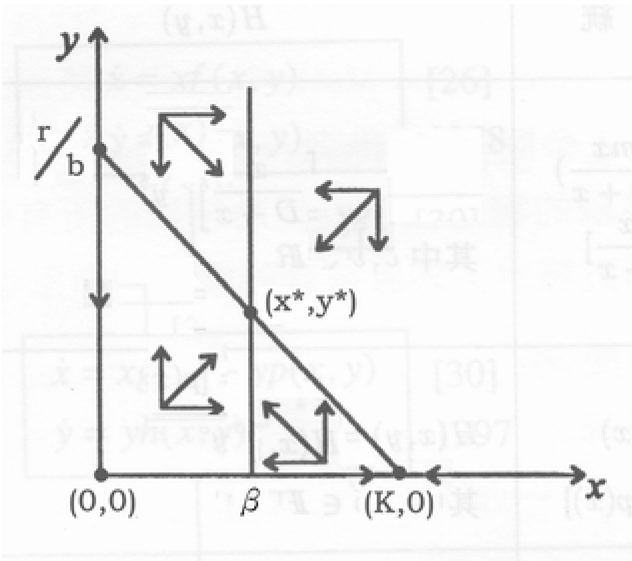

Figure 11.3

Remark:

There are five distinct types of biological interactions between two species:

1. **Mutualism or symbiosis (++):** Each species has a positive effect on the other.

2. **Competition (--):** Each species has a negative effect on the other.

3. **Commensalism (+0):** One species benefits from the interaction, whereas the other is unaffected.

4. **Amensalism (-0):** One species is negatively affected, whereas the other is unaffected.
5. **Predation (+0):** One species benefits, whereas the other is negatively affected.