§3.9 Two dimensional linear flow

A two-dimensional linear system is a system of the form

\[ \begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*} \tag{9.1} \]

where \(a, b, c, d\) are parameters. (9.1) can be written in vector form

\[ \dot{X} = AX \]

where

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \tag{9.2} \]

Such a system is a linear in the sense that if \( X_1(t) \) and \( X_2(t) \) are solutions of (9.2), then is any linear combination \( c_1X_1(t) + c_2X_2(t) \). Assume \( \det A \neq 0 \), then \( X = 0 \) is the unique equilibrium of (9.2). The solution of \( \dot{X} = AX \) can be visualized as trajectories moving on the \((x, y)\) plane, in this context called phase plane.

For the general solutions of (9.2) we seek trajectories of the form

\[ X(t) = e^{\lambda t}v \tag{9.3} \]

where \( v \neq 0 \) is some fixed vector to be determined and \( \lambda \in C \) to be determined. Substitute (9.3) into (9.2), we obtain \( \lambda e^{\lambda t}v = e^{\lambda t}Av \). Cancelling the nonzero scalar factor \( e^{\lambda t} \) yields

\[ Av = \lambda v \tag{9.4} \]

i.e. \((\lambda, v)\) is an eigenpair of \(2 \times 2\) matrix \(A\). It is easy to find that the eigenvalues \(\lambda_1, \lambda_2\) of \(A\) are

\[ \lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{9.5} \]

where

\[ \tau = \text{trace}(A) = a + d, \quad \Delta = \det A = ad - bc \]

If \(\lambda_1 \neq \lambda_2\) then the corresponding eigenvectors \(v_1\) and \(v_2\) are linearly independent. Any initial condition \(X_0\) can be written as a linear combination of eigenvectors, say,
$X_0 = c_1 V_1 + c_2 V_2$. This observation allows us to write down the general solution $X(t)$ as

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

(9.6)

There are following cases for various $\lambda_1$ and $\lambda_2$.

**Case 1:** $\lambda_1$, $\lambda_2$ are real and $\lambda_2 < \lambda_1$

**Case 1a** (Stable node) $\lambda_2 < \lambda_1 < 0$

Let $L_1$, $L_2$ be the line generated by $V_1$, $V_2$ respectively. Since $\lambda_2 < \lambda_1 < 0$,

$X(t) \approx c_1 e^{\lambda_1 t} V_1$ as $t \to \infty$ and the trajectories are tangent to $L_1$

**Case 1b** (Unstable node) $0 < \lambda_2 < \lambda_1$

Then $X(t) \approx c_1 e^{\lambda_1 t} V_1$ as $t \to \infty$
Case 1c (Saddle point) $\lambda_2 < 0 < \lambda_1$. In this case, the origin 0 is called a saddle point and $L_1$, $L_2$ are unstable manifold and stable manifold.

Case 2: $\lambda_1, \lambda_2$ are complex.

Let $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and $V_1 = U + iV$ and $V_2 = U + iV$ be complex eigenvectors. Then

$$x(t) = ce^{i(\alpha - i\beta)t}V_1 + \overline{c}e^{i(\alpha + i\beta)t}V_1 = 2\text{Re}(ce^{i(\alpha + i\beta)t}V_1)$$

Let $c = ae^{i\delta}$. Then
\[ x(t) = 2ae^{\alpha t} (u \cos(\beta t + \delta) - v \sin(\beta t + \delta)) \].

Let \( P \) and \( Q \) be the line generated by \( U \), \( V \) respectively.

**Case 2a (Center)** \( \alpha = 0 \), \( \beta \neq 0 \).

**Case 2b (Stable focus, spiral)** \( \alpha < 0 \), \( \beta \neq 0 \).

**Case 2c (Unstable focus, spiral)** \( \alpha > 0 \), \( \beta \neq 0 \).