§3.9 Two dimensional linear flow

A two-dimensional linear system is a system of the form

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned} \qquad (9.1)$$

where a, $,b$, c, d are parameters. (9.1) can be written in vector form

$$\dot{X} = AX$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (9.2)$$

Such a system is a linear in the sense that if $X_1(t)$ and $X_2(t)$ are solutions of (9.2), then is any linear combination $c_1X_1(t) + c_2X_2(t)$. Assume $\det A \neq 0$, then $X = 0$ is the unique equilibrium of (9.2). The solution of $\dot{X} = AX$ can be visualized as trajectories moving on the (x, y) plane, in this context called phase plane.

For the general solutions of (9.2) we seek trajectories of the form

$$X(t) = e^{\lambda t} v \qquad (9.3)$$

where $v \neq 0$ is some fixed vector to be determined and $\lambda \in C$ to be determined. Substitute (9.3) into (9.2), we obtain $\lambda e^{\lambda t} v = e^{\lambda t} Av$. Cancelling the nonzero scalar factor $e^{\lambda t}$ yields

$$Av = \lambda v \qquad (9.4)$$

i.e. (λ, v) is an eigenpair of 2×2 matrix A. It is easy to find that the eigenvalues λ_1, λ_2 of A are

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \qquad (9.5)$$

where

$$\tau = trace(A) = a + d, \quad \Delta = \det A = ad - bc$$

If $\lambda_1 \neq \lambda_2$ then the corresponding eigenvectors v_1 and v_2 are linearly independent. Any initial condition X_0 can be written as a linear combination of eigenvectors, say,

$X_0 = c_1V_1 + c_2V_2$. This observation allows us to write down the general solution $X(t)$ as

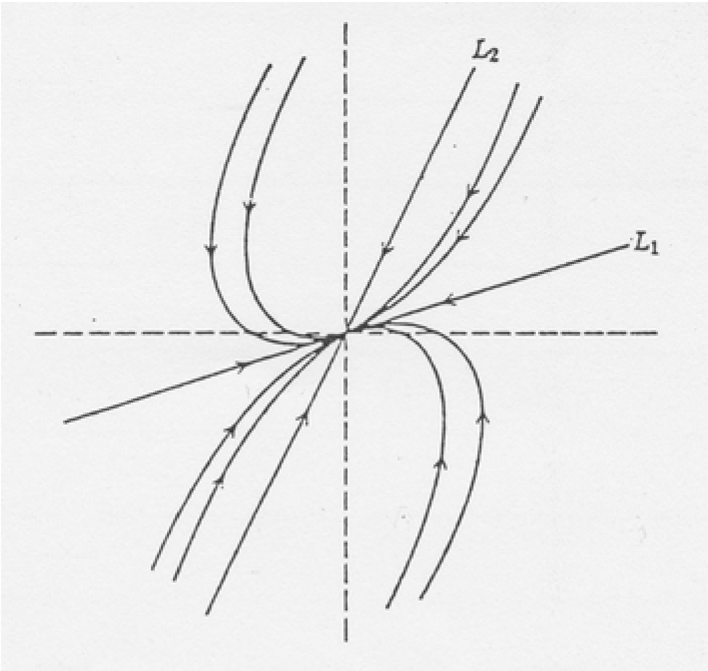
$$X(t) = c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2 \tag{9.6}$$

There are following cases for various λ_1 and λ_2.

Case 1: λ_1, λ_2 are real and $\lambda_2 < \lambda_1$

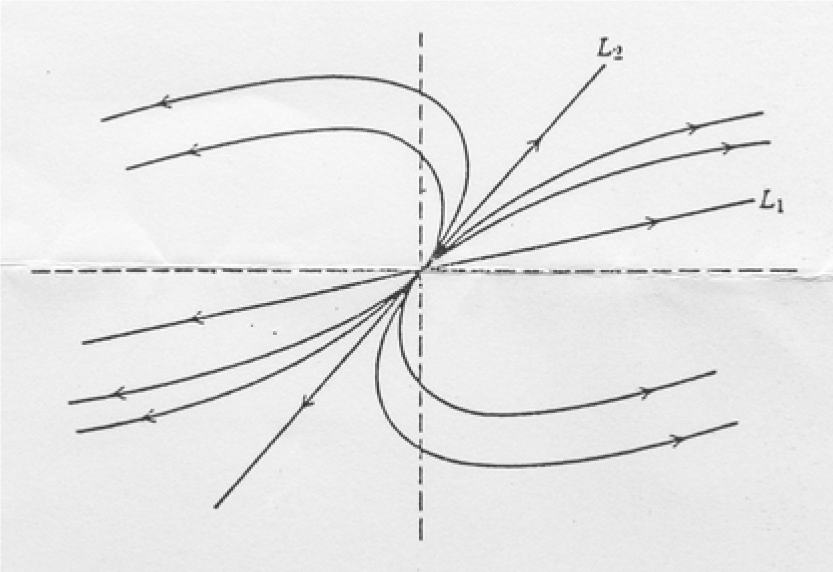
Case 1a (Stable node) $\lambda_2 < \lambda_1 < 0$

Let L_1, L_2 be the line generated by V_1, V_2 respectively. Since $\lambda_2 < \lambda_1 < 0$, $X(t) \approx c_1e^{\lambda_1 t}V_1$ as $t \to \infty$ and the trajectories are tangent to L_1

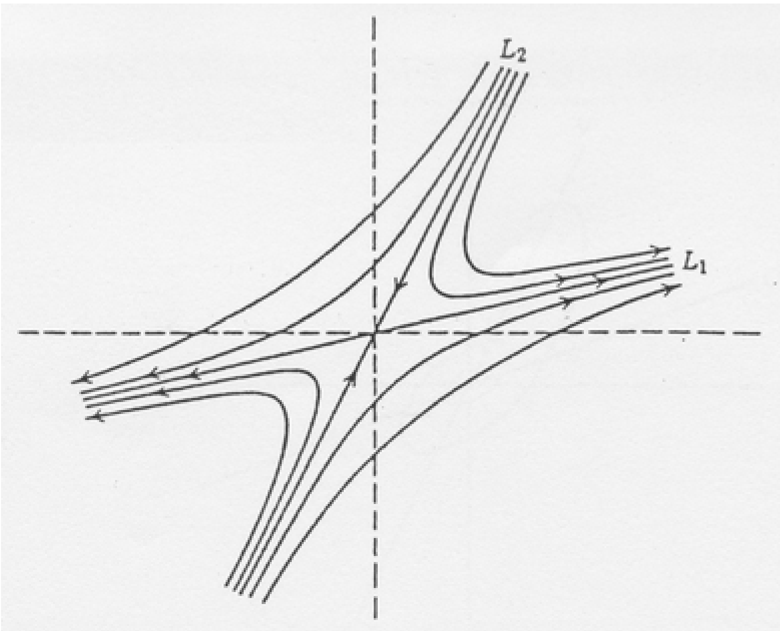


Case 1b (Unstable node) $0 < \lambda_2 < \lambda_1$

Then $X(t) \approx c_1e^{\lambda_1 t}V_1$ as $t \to \infty$



Case 1c (Saddle point) $\lambda_2 < 0 < \lambda_1$. In this case, the origin 0 is called a saddle point and L_1, L_2 are unstable manifold and stable manifold.



Case 2: λ_1, λ_2 are complex.

Let $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and $V_1 = U + iV$ and $V_2 = U + iV$ be complex eigenvectors. Then

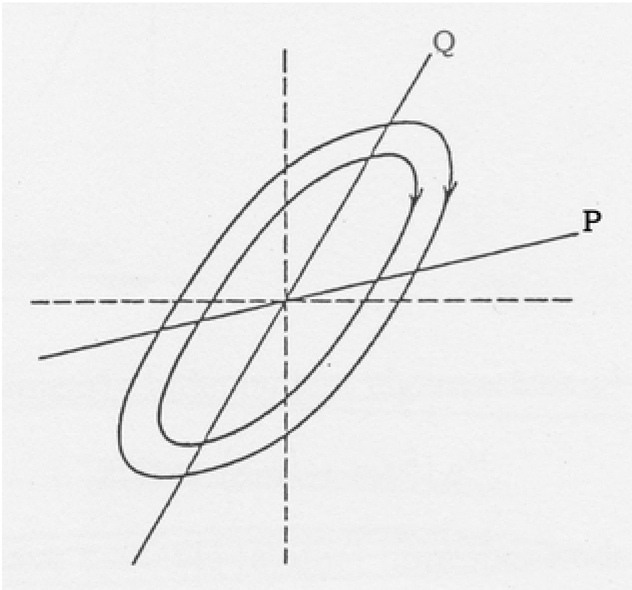
$$x(t) = ce^{(\alpha+i\beta)t}V_1 + \bar{c}e^{(\alpha-i\beta)t}\overline{V_1} = 2\,\mathrm{Re}(ce^{(\alpha+i\beta)t}V_1)$$

Let $c = ae^{i\delta}$. Then

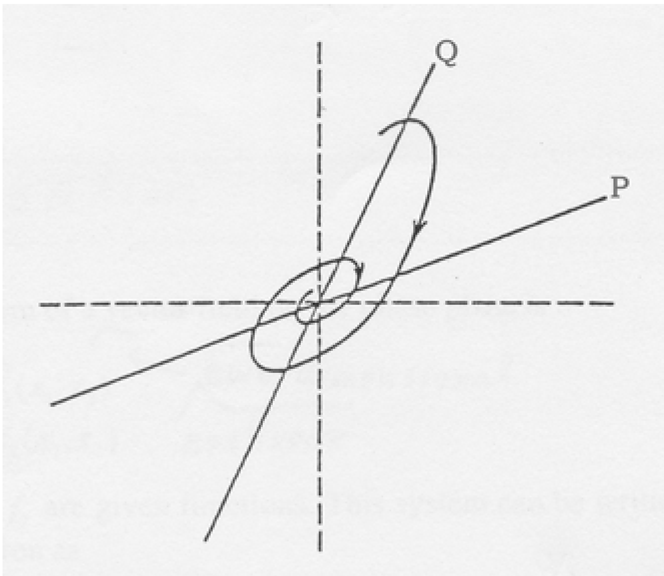
$$x(t) = 2ae^{\alpha t}(u\cos(\beta t + \delta) - v\sin(\beta t + \delta)).$$

Let P and Q be the line generated by U, V respectively.

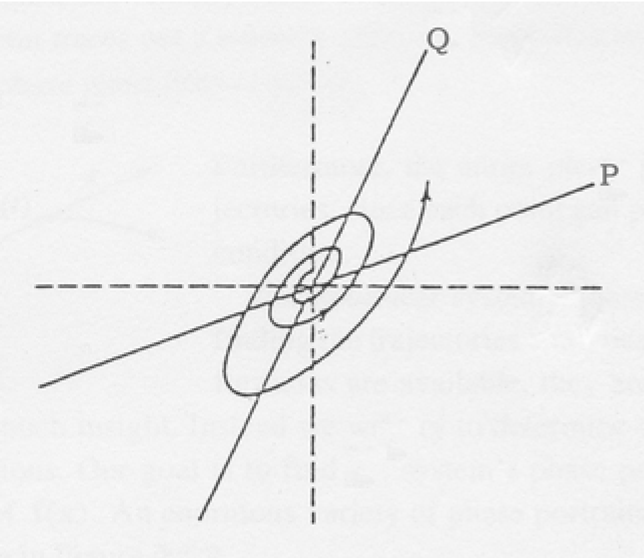
Case 2a (Center) $\alpha = 0$, $\beta \neq 0$.



Case 2b (Stable focus, spiral) $\alpha < 0$, $\beta \neq 0$.



Case 2c (Unstable focus, spiral) $\alpha > 0$, $\beta \neq 0$.


Q
P