§3.10 Two dimensional phase portraits, Fixed points and linearization

The general form of a nonlinear two dimensional vector filed on the phase plane is

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 given functions. This system can be written more compactly in vector notation as

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x})$$

where $\mathrm{x} = (x_1, x_2)$ and $\mathrm{f}(\mathrm{x}) = (f_1(\mathrm{x}), f_2(\mathrm{x}))$. Here x represents a point in the phase plane, and $\dot{\mathrm{x}}$ is the velocity vector at that point. By flowing along the vector field, a phase point traces out a solution $\mathrm{x}(t)$, corresponding to a trajectory winding through the phase plane (Figure 3.10.1).

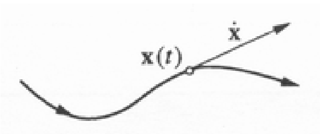

Figure 3.10.1

Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition.

For nonlinear systems, there's typically no hope of finding the trajectories analytically. Even when explicit formulas are available, they are often too complicated to provide much insight. Instead we will try to determine the *qualitative* behavior of the solutions. Our goal is to find the system's phase portrait directly from the properties of $\mathrm{f}(\mathrm{x})$. An enormous variety of phase portraits is possible; one example is shown in Figure 3.10.2.

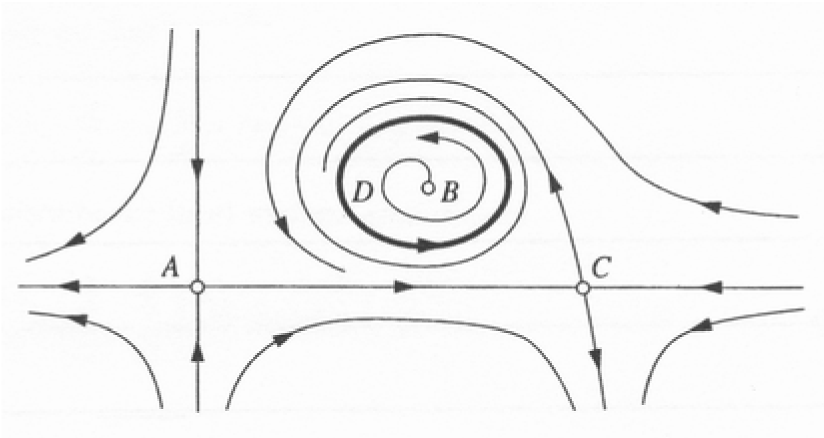

Figure 3.10.2

Some of the most salient features of any phase portrait are:

1. The ***fixed points***, like A, B, and C in Figure 3.10.2. Fixed points satisfy $\mathrm{f}(\mathrm{x}) = 0$, and correspond to steady states or equilibria of the system.
2. The ***closed orbits***, like D in Figure 3.10.2. These correspond to periodic solutions, i.e., solutions for which $\mathrm{x}(t+T) = \mathrm{x}(t)$ for all t, for some $T > 0$.
3. The arrangement of trajectories near the fixed points and closed orbits. For example, the flow pattern near A and C is similar, and different from that near B.
4. The stability or instability of the fixed points and closed orbits. Here, the fixed points A, B, and C are unstable, because nearby trajectories tend to move away from them, whereas the closed orbit is stable D.

In this section we extend the ***linearization*** technique developed earlier for one-dimensional systems. The hope is that we can approximate the phase portrait near a fixed point by that of a corresponding linear system.

Linearized System

Consider the system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is an equilibrium, i.e.,

$$f(x^*, y^*) = 0, \quad g(x^*, y^*) = 0.$$

Let

$$u = x - x^*, \quad v = y - y^*$$

denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. Let's do the u –equation first:

$$\begin{aligned}
\dot{u} &= \dot{x} && \text{(since } x^* \text{ is a constant)} \\
&= f(x^* + u, y^* + v) && \text{(by substitution)} \\
&= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(Taylor series expansion)} \\
&= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(since } f(x^*, y^*) = 0\text{).}
\end{aligned}$$

To simplify the notation, we have written $\partial f/\partial x$ and $\partial f/\partial y$, but please remember that these partial derivatives are to be evaluated *at the fixed point* (x^*, y^*); thus they are *numbers*, not functions. Also, the shorthand notation $O(u^2, v^2, uv)$ denotes ***quadratic terms*** in u and v. Since u and v are small, these quadratic terms are *extremely* small.

Similarly we find

$$\dot{v} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv).$$

Hence the disturbance (u, v) evolves according to

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}+\text{Quadratic terms.} \tag{10.1}$$

The matrix

$$J(x^*,y^*)=\begin{pmatrix}\dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y}\end{pmatrix}_{(x^*,y^*)}$$

is called ***Jacobian matrix*** at the fixed point (x^*,y^*).

Now since the quadratic terms in (10.1) are tiny, it's tempting to neglect them altogether. If we do that, we obtain the ***linearized system***

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \tag{10.2}$$

whose dynamics can be analyzed by the methods of Section 3.9.

The Effect of Small Nonlinear Terms

Is it really safe to neglect the quadratic terms in (10.1)? In other words, does the linearized system give a qualitatively correct picture of the phase portrait near (x^*,y^*)? The answer is *yes*, if the linearized system predicts a saddle, node, or a spiral, then the fixed point *really is* a saddle, node, or spiral for the original nonlinear system. The equilibrium (x^*,y^*) is locally stable if each eigenvalues λ of $J(x^*,y^*)$ satisfies $\operatorname{Re}\lambda<0$.

Remark: Consider the map

$$x_{n+1}=f(x_n,y_n)$$

$$y_{n+1} = g(x_n, y_n)$$

and suppose (x^*, y^*) is a fixed point, i.e.

$$f(x^*, y^*) = x^*, \quad g(x^*, y^*) = y^*$$

Let $u_n = x_n - x^*$, $v_n = y_n - y^*$ denote the components of a small disturbance from the fixed point. Then

$$\begin{aligned} u_{n+1} &= x_{n+1} - x^* = f(x_n, y_n) - f(x^*, y^*) \\ &= (x_n - x^*)\tfrac{\partial f}{\partial x}(x^*, y^*) + (y_n - y^*)\tfrac{\partial f}{\partial y}(x^*, y^*) + \text{H.O.T.} \\ &= u_n \tfrac{\partial f}{\partial x} + v \tfrac{\partial f}{\partial y} + \text{H.O.T.} \end{aligned}$$

Similary

$$v_{n+1} = u_n \frac{\partial g}{\partial x}(x^*, y^*) + v_n \frac{\partial g}{\partial y}(x^*, y^*) + \text{H.O.T.}$$

Then the deviations (u_n, v_n) evolves according to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x,y)=(x^*,y^*)} + \text{H.O.T.}$$

We obtain the linearized system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x,y)=(x^*,y^*)} \begin{bmatrix} u_n \\ v_n \end{bmatrix} = J(x^*, y^*) \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

The fixed point (x^*, y^*) is locally stable if each eigenvalue λ of jacobian matrix $J(x^*, y^*)$ satisfies $|\lambda| < 1$.