§1.2 Single Population Growth : Discrete case

(i) Malthusim Model

$$N(t+1) = \lambda N(t) \quad , \quad N_{k+1} = \lambda N_k$$

$$\lambda = b - d = \text{birth rate} - \text{death rate}.$$

Then we have

$$N_k = \lambda^k N_0$$

$0 < \lambda < 1 \Rightarrow N_k \to 0$ as $k \to \infty$. If $\lambda > 1$ then $N_k \to \infty$

(ii) Discretized logistic equation

$$\frac{dN}{dt} \approx \frac{N(t+\Delta t) - N(t)}{\Delta t} = r N(t) \left(1 - \frac{N(t)}{K}\right)$$

$$\therefore N(t+\Delta t) = N(t) + r \Delta t N(t) \left(1 - \frac{N(t)}{K}\right)$$

$$= (r\Delta t + 1) N(t) - \frac{r\Delta t}{K} (N(t))^2$$

$$N_{k+1} = \tilde{r} N_k \left(1 - \frac{N_k}{\tilde{K}}\right)$$

$$\tilde{r} = r\Delta t + 1 \ , \ \tilde{K} = \frac{r\Delta t + 1}{\left(\frac{r\Delta t}{K}\right)}$$

Per-Duet

Scaling: $\tilde{N}_k = \dfrac{N_k}{\tilde{K}}$

$$\tilde{N}_{k+1} = \tilde{r}\,\tilde{N}_k\,(1-\tilde{N}_k)$$

Thus we obtain discretize logistic equation

$$x_{k+1} = r\,x_k(1-x_k) = f(x_k)$$

We note that

$$f: [0,1] \longrightarrow [0,1] \quad \text{if } 0 < r \le 4$$

Definition: We say x^* is a fixed point $\underbrace{}_{\text{of } x_{k+1}=f(x_k)}$ if

$$x^* = f(x^*)$$

We say x^* is a n-periodic point if

$$f^{(n)}(x^*) = x^*$$

i.e. x^* is a fixed point of $f^n(x) \overset{\text{def}}{=} f(f \cdots f(x))$

Analysis:

Consider the map

$$x_{k+1} = f(x_k)$$

Per-Duet

2 **Step 1**: Find all ~~equil~~ fixed points

Step 2: Local stability analysis about $x = x^*$

Let

$z_k = x_k - x^*$

$$z_{k+1} = x_{k+1} - x^* = f(x_k) - x^* = f(x_k) - f(x^*)$$

(Taylor ~~Expansion~~) $= f'(x^*) \cdot (x_k - x_k) +$ Higher order term

$= f'(x^*)\, z_k +$ Higher order term

Consider linearized map:

$$z_{k+1} = f'(x^*)\, z_k$$

$$\Rightarrow \quad z_k = \left(f'(x^*)\right)^k z_0$$

Hence

if $|f'(x^*)| < 1$ then $z_k \to 0$ as $k \to \infty$

if $|f'(x^*)| > 1$ then $|z_k| \to \infty$ as $k \to \infty$

Thus

$|f'(x^*)| < 1 \Longrightarrow x^*$ is locally stable i.e. if x_0 is near x^* then $\lim_{k\to\infty} x_k = x^*$

$|f'(x^*)| > 1 \implies x^*$ is unstable.

i.e there exist x_0 near x^*, but x_k, k large, escape nbd of x^*.

Analysis of discretized logistic equation:

$$x_{k+1} = f(x_k) = r\,x_k(1-x_k), \quad 0<r<4, \quad x_k \in [0,1]$$

Step 1: Find all fixed points

Solve $x = f(x) = r\,x(1-x)$

$$x\,[1 - r(1-x)] = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = \frac{r-1}{r} > 0 \iff r > 1$$

Hence

If $0<r<1$, $x^*=0$ is the only fixed pt.

$$f'(x) = r(1-2x)$$

$$|f'(0)| = |r| = r < 1 \text{ if } 0<r<1$$

Hence $x^*=0$ is locally stable

If $r>1$, then the ~~fixd~~ fixed point $x^* = \frac{r-1}{r} > 0$ exists.

$$f'(x^*) = r\cdot(1-2x^*) = r\cdot\left(1-\frac{2(r-1)}{r}\right) = -r+2$$

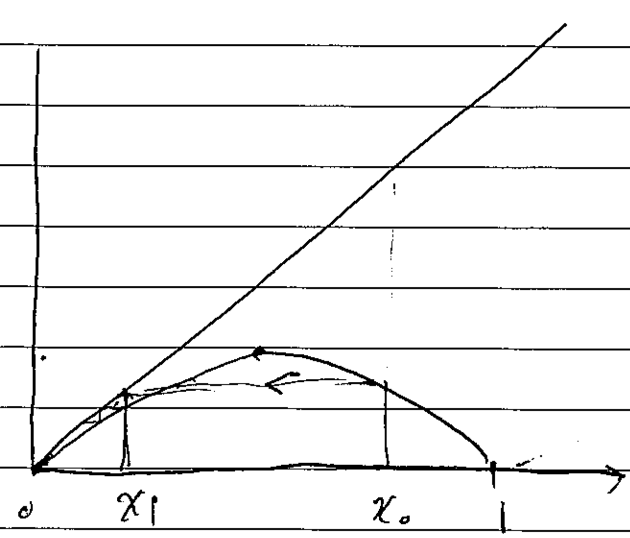
$$|f'(x^*)| = |-r+2| = |r-2| < 1 \iff -1 < r-2 < 1$$

$$\iff 1 < r < 3$$

Hence the fixed point x^* is locally stable if $1 < r < 3$ which the fixed point 0 becomes unstable when $r > 1$

<u>Definition</u>: We say x^* is a n-periodic point of the map $f(x)$ if $f^{(n)}(x^*) = x^*$ where

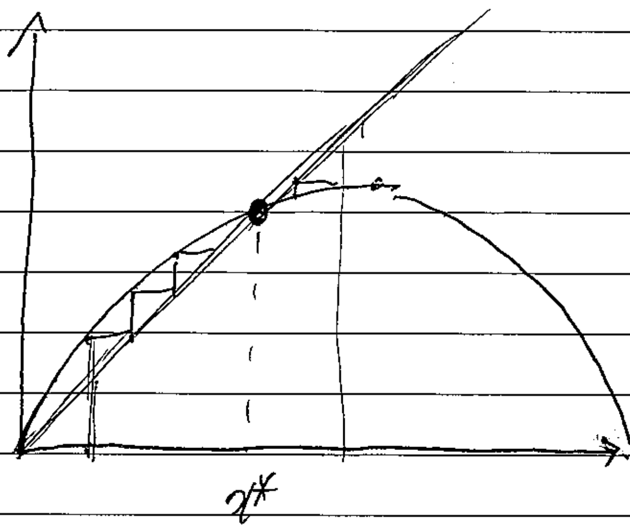
$$f^{(n)}(x) = \underbrace{f(f(\cdots(f(x))}_{n \text{ times composition}}$$



$0 < r < 1$

$$\lim_{k\to\infty} x_k = 0$$

Per-Duet



$1 < r < 3$

$\lim_{k\to\infty} x_k = x^*$

Example Show that the logistic map has 2-cycle for all $r > 3$

$f(x) = rx(1-x)$

A 2 cycle exists if and only if there are two points p and q such that $f(p) = q$, $f(q) = p$

Solve $f^2(x) = f(f(x)) = x$

Obviously $x^* = 0$, $x^* = 1 - \frac{1}{r}$ are trivial solutions of $f^2(x^*) = x^*$ $\because$ $f(x^*) = x^*$

$f^2(x) - x = 0 \Rightarrow r^2 x(1-x)\left[1 - rx(1-x)\right] - x = 0$

$\Rightarrow p, q = \dfrac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$

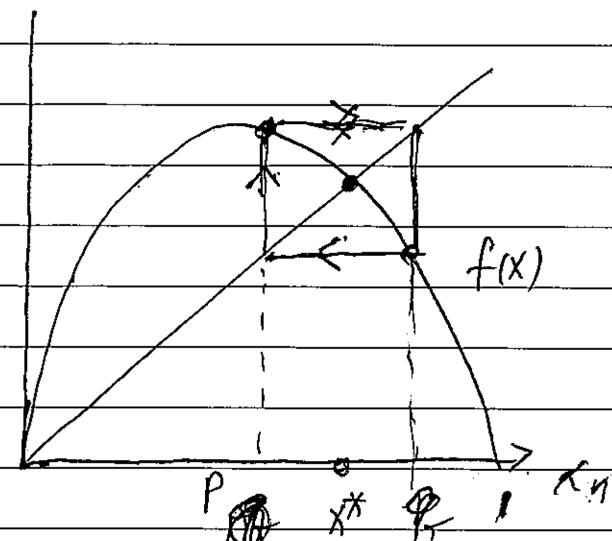
$\therefore$ 2-cycle exists for $r > 3$

$r^2(1-x)\left[rx^2 - rx + 1\right] - 1 = 0$

Per-Duet

$$(x-1)(rx^2-rx+1)+\frac{1}{r^2}=0$$

$$rx^3-2rx^2+(r+1)x+\left(\frac{1}{r^2}-1\right)=0 \Rightarrow \left(x-\frac{r-1}{r}\right)\left(rx^2-(r+1)x \right.$$
$$\left. \left(x+\left(\frac{1}{r}-1\right)\right)+\left(\frac{1}{r}+1\right)\right]$$



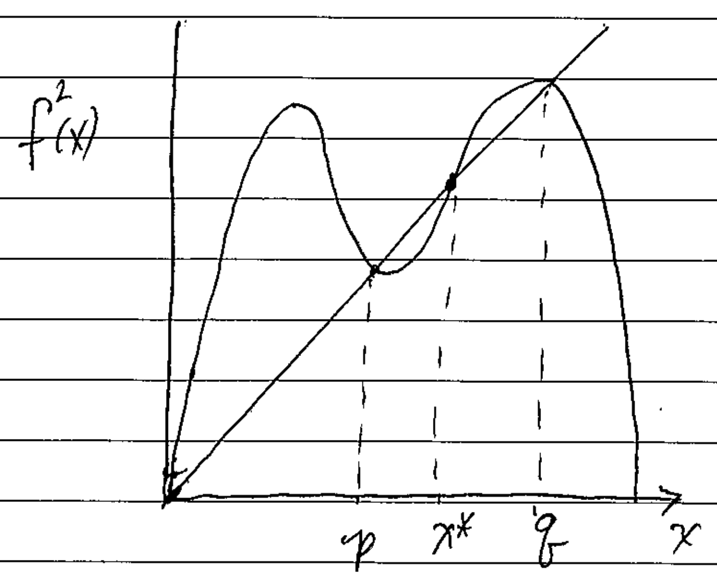
Hence

$$(x-p)(x-q)$$

$$=x^2-\frac{r+1}{r}x+\left(\frac{1}{r}+1\right)\frac{1}{r}$$

$$p+q=\frac{r+1}{r}$$

$$pq=\frac{1}{r}\left(\frac{1}{r}+1\right)$$



Example: Show that the 2-cycle is stable for

$$3<r<1+\sqrt{6}=3.449\ldots$$

Let $\lambda=\frac{d}{dx}(f(f(x)))\Big|_{x=p}=f'(f(p))\,f'(p)=f'(q)\,f'(p)$

$$\lambda=r(1-2q)\cdot r(1-2p)$$
$$=r^2\left[1-2(p+q)+4pq\right]$$

$$= r^2 \left[1 - \frac{2(r+1)}{r} + 4\frac{(r+1)}{r^2} \right]$$

$$= 4 + 2r - r^2$$

$$|4+2r-r^2| < 1 \iff 3 < r < 1+\sqrt{6}$$

<u>Example</u>: Ricker's equation which has been applied extensively but it was originally developed for fish population based on stock and recruitment.

$$x_{k+1} = x_k \exp\left(r\left(1-\frac{x_k}{K}\right)\right)$$

$$= a x_k e^{-\alpha x_k} = f(x_k), \quad a>0, \ \alpha>0$$

$$f: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

(i) Find fixed pt of $f(x)$

(ii) Do stability analysis

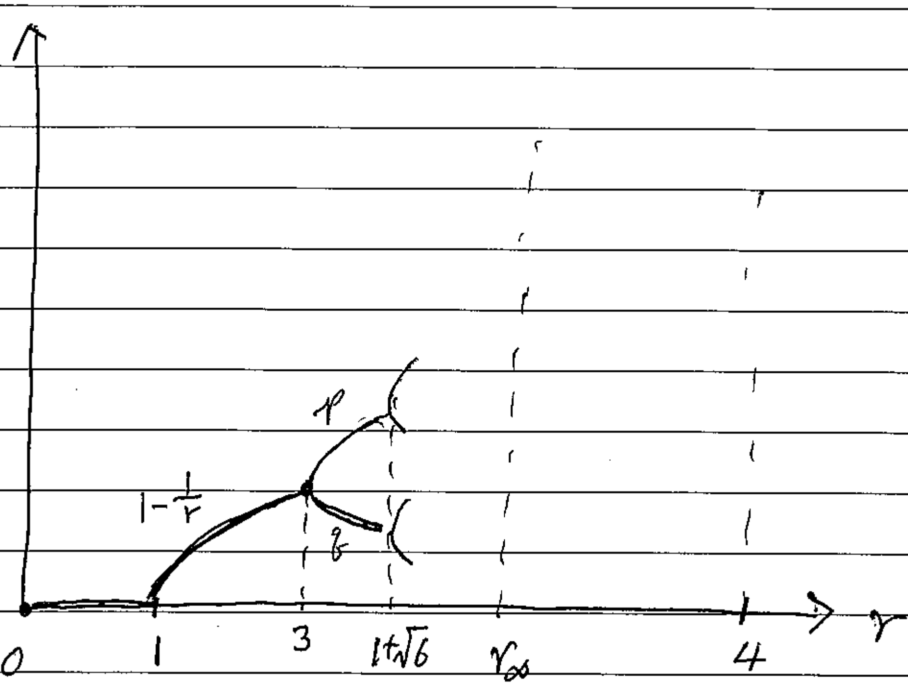
(iii) Find 2-cycle solution & Do the stability analysis of 2-cycle

(iv) plot orbit diagram with $\alpha = 1$, $a > 0$

* period three implies chaos L. Y. Li & Yorke

* Strogatz: Nonlinear Dynamics and Chaos, Chapter 10

<u>orbit diagram</u>:



$r_0 = 1 < r < 3$, x_k converges to fixed pt x^*

$r_1 = 3 \le r < 1+\sqrt{6} \approx 3.449 = r_2$ period 2

$r_2 \le r < 3.54409 = r_3$ period $4 = 2^2$

$r_3 \le r < r_4 = 3.5644$ period $8 = 2^3$

⋮

Logistic map exhibits a route to chaos by <u>period doubling</u>

Q: $r > r_\infty$? It is very complicated.

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta = 4.669$$, δ is called universal number discovered by Feigenbaum. (Renormalization)