proof for second derivative test:

Recall that Taylor thm for two variables

$f(a+h, b+k) - f(a,b)$

$= f_x(a,b)h + f_y(a,b)k + \frac{1}{2}\left(h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right) + \underbrace{\varepsilon \rho^2}$

where $\rho^2 =: h^2+k^2$, $\varepsilon \to 0$ as $\rho \to 0$

Remainder

Now from assumption: $\nabla f(a,b) = 0$

$\therefore f(a+h, b+k) - f(a,b) = \frac{1}{2}\underbrace{(h^2A + 2hkB + k^2C)}_{\text{quadratic form}} + \varepsilon\rho^2$

Denote $Q(h,k) = Ah^2 + 2B\,hk + Ck^2$

(i) Q is definite i.e. Q is of one sign only $\forall\, h,k$ and vanishes only for $h=0=k$.
We say it's positive definite or negative definite according to this sign
eg: $Q = h^2+k^2$ is positive definite b/c $Q > 0$ $\forall\, h,k \neq 0$, and $= 0$ if $h=0=k$

eg: $Q = -h^2 + 2hk - 2k^2 = -(h-k)^2 - k^2$ is negative definite b/c $Q < 0$ $\forall\ h, k \neq 0$ and $= 0$ if $h = k = 0$

(ii) Q is indefinite i.e. Q can have different sign according to different h, k

eg: $Q = 2hk$, then $Q > 0$ if h, k same sign, $Q < 0$ if opposite sign

(iii) Q is called semi-definite if $Q = 0$ for pts other than $(0,0)$, but for any $Q \neq 0$, it's of one sign.

eg: $Q = (h+k)^2$, then $Q = 0$ $\forall\ h = -k$
$Q > 0$ otherwise

(i) Clearly, this can happen (ie Q is of one sign) if and only if $AC - B^2 > 0$
and is positive def if $A > 0$
negative def if $A < 0$

$$\because\ Q = A\left[\underbrace{\left(h + \frac{B}{A}k\right)^2}_{\geq 0} + \frac{AC - B^2}{A^2}\underbrace{k^2}_{\geq 0}\right] \quad (*)$$

$\therefore$ if $AC - B^2 > 0$ then Q is of one sign and also $A>0 \Rightarrow Q>0$

$A<0 \Rightarrow Q<0$

(ii) In order to have indefinite, we see that we need $AC-B^2<0$ (from look at $(*)$ above)

(iii) In order to have semi-definite, if $AC-B^2=0$, then $h=-\frac{B}{A}k$ will give us $Q=0$ and for any other h,k Q will be of one sign (depending solely on A).

We will show (i) $\Rightarrow$ local min, max

(ii) $\Rightarrow$ saddle pt

(iii) $\Rightarrow$ Not known

(i) if Q is positive definite, then $\exists\ m>0$ (indep. of h & k) s.t.

$Q \geq 2m(h^2+k^2)$

b/c, if we consider the fun $\bar{Q}(u,v)=\frac{Q(h,k)}{h^2+k^2}$

with $u=\frac{h}{\sqrt{h^2+k^2}}$, $v=\frac{k}{\sqrt{h^2+k^2}}$, Then $u^2+v^2=1$

and $\overline{Q}(u,v) = Au^2 + 2Buv + Cv^2$, and on the set $u^2+v^2=1$, $\overline{Q}$ must have a min b/c $u^2+v^2=1$ is a closed set, call it $2m$, then $\overline{Q} \geq 2m$. Moreover $2m \neq 0$ b/c if $2m = 0 \Rightarrow \overline{Q} = 0 \Rightarrow u = 0 = v$ but $u^2+v^2=1$ so impossible.

Thus $\overline{Q} \geq 2m > 0 \quad \therefore Q \geq 2m\underbrace{(h^2+k^2)}_{>0}$

Therefore $f(a+h, b+k) - f(a,b)$

$$= \frac{1}{2}Q(h,k) + \varepsilon\rho^2 \geq (m+\varepsilon)\rho^2$$

and since $\varepsilon \to 0$ as $\rho^2 \to 0$ we can choose ρ small enough s.t. $|\varepsilon| < \frac{1}{2}m$

$\Rightarrow f(a+h, b+k) - f(a,b) \geq \frac{1}{2}m\rho^2 > 0$

$\Rightarrow f(a,b)$ local min in a small nbd of (a,b) if Q positive def.

Similarly, $f(a,b)$ local max in a small nbd if Q is negative definite.

(ii) if the form is indefinite, then there's a pair (h_1, k_1) for which $Q < 0$ and a pair (h_2, k_2) for which $Q > 0$

$\therefore$ we can find $m > 0$ s.t.

$Q(h_1, k_1) < -2m\rho_1^2 = -2m(h_1^2 + k_1^2)$

$Q(h_2, k_2) > 2m\rho_2^2 = 2m(h_2^2 + k_2^2)$

let $h = th_1$, $k = tk_1$, $\rho = h^2 + k^2$,

then $Q(h,k) = t^2(Q_1, k_1)$, $\rho^2 = t^2\rho_1^2$

i.e. we consider the line segment from (a, b) to $(a+h_1, b+k_1)$

($\because$ any pts in between is $(a+th_1, b+tk_1)$)

so clearly $Q(h,k) < -2m\rho^2$ for such (h,k)

so as before, we get

$f(a+h, b-k) - f(a,b) < -m\dfrac{\rho^2}{2}$

for (h,k) small enough on the line segment ($\because$ choose t small enough, as the ε in (i))

$\therefore$ $f(a+h, b+k) < f(a,b)$ for those h, k

Similarly, we do this for
$h = th_2$, $k = tk_2$ and show
that $f(a+h, b+k) > f(a,b)$ for those
h, k small enough on the line
segment

$\Rightarrow$ In a small nbd of (a,b) we have
$f(x,y) > f(a,b)$ & $f(x,y) < f(a,b)$

$\Rightarrow$ $(a, b, f(a,b))$ is a saddle pt!