P11

2024年3月4日　　下午 04:03

Now let's look at the proofs :

① If it conv, then $S_n := \sum_{k=1}^{n} a_k \to L$, also $S_{n-1} := \sum_{k=1}^{n-1} a_k \to L$

$a_n = S_n - S_{n-1}$ and since both limit exist , the limit of their sum / difference also exists $\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$

Alternatively, one can also prove it as follows

If $\lim_{n\to\infty} a_n = c \gtrneqq 0 \Rightarrow$ given $0 < \varepsilon < L$, for $k > k_0$, $|a_k - c| < \varepsilon$, i.e. $0 < c - \varepsilon < a_k < c + \varepsilon$

$$\Rightarrow \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{k_0} a_k + \sum_{k=k_0+1}^{\infty} a_k \geq \sum_{k=1}^{k_0} a_k + \sum_{k=k_0+1}^{\infty} (c - \varepsilon) = \infty$$

$\divideontimes$

Similar proof for $c \lneqq 0$, Thus $\lim_{n\to\infty} a_n$ must be zero.

#

② $S_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$, $W_n := S_n + t_n = \sum_{k=1}^{n} (a_k + b_k)$

$\therefore \lim_{n\to\infty} W_n = \lim_{n\to\infty} (S_n + t_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} t_n$ by property for limits for sequences.

③ to prove this, we need a Lemma :

Lemma : A series $\sum_{k=1}^{\infty} a_k$ with nonnegative terms converges iff the sequence of partial sum $S_n = \sum_{k=1}^{n} a_k$ is bounded.

Pf : if $\sum_{k=1}^{\infty} a_k$ conv. then by def it means $\lim_{n\to\infty} S_n$ converges, thus S_n is bounded ($\because$ a conv. sequence must be bounded)

Conversely, if S_n is bounded, then since S_n is non-decreasing ($\because a_k$ nonnegative) $\Rightarrow \lim_{n\to\infty} S_n$ conv.

$\Rightarrow \sum_{k=1}^{\infty} a_k$ conv. ($\because$ monotonic sequence thm)　#

P12

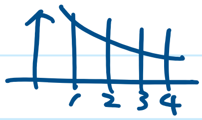
2024年3月4日　　下午 01:42

Next, let's consider a sequence $b_n := \int_1^n f(x)\,dx$, $n=1,\cdots$

If f is cont., positive, and decreasing on $[1,\infty)$, then

$\int_1^\infty f(x)\,dx$ converges iff $\{b_n\}_{n=1}^\infty$ converges. Ex 69, §11.7

(proof see next page)

Moreover, since f is decreasing $\Rightarrow$ 

$\therefore$ $f(2)+\cdots+f(n) \le \underbrace{\int_1^n f(x)\,dx}_{b_n} \le f(1)+\cdots+f(n-1)$

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If $\int_1^\infty f(x)\,dx$ converges, then $\{b_n\}_{n=1}^\infty$ conv., then $\{b_n\}_{n=1}^\infty$ is bounded

$\Rightarrow$ $f(2)+\cdots+f(n)$ is bounded

$\Rightarrow$ $S_n = \sum_{k=1}^n a_k = f(1)+\cdots+f(n)$ is bounded

$\Rightarrow$ $\sum_{k=1}^n f(k)$ conv. by the Lemma, b/c $a_k = f(k)$ nonnegative

To prove "if $\sum_{k=1}^\infty a_k$ conv then $\int_1^\infty f(x)\,dx$ conv." we prove by contradiction:

Suppose $\int_1^\infty f(x)\,dx$ div. then $\{b_n\}_{n=1}^\infty$ div. Moreover, since $f>0$,

b_n is increasing $\Rightarrow$ b_n must be unbounded ($\because$ if it's bounded,

then by monotonic seq. thm, $\{b_n\}$ will conv.)

$\Rightarrow$ $f(1)+\cdots+f(n-1)$ is also unbounded

$\Rightarrow$ $S_n = \sum_{k=1}^n a_k = f(1)+\cdots+f(n)$ is unbounded.

$\Rightarrow$ $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty f(k)$ div $\rightarrow\!\leftarrow$ thus $\int_1^\infty f(x)\,dx$ must conv.

#

Ex 69 from section 11.7 improper integrals

Lemma: If f is cont, positive, non-decreasing on $[1,\infty)$, then $\int_1^\infty f(x)\,dx$ conv. iff $\{b_n := \int_1^n f(x)\,dx\}$ conv.

Pf: Since f is cont & positive, $F(t) := \int_1^t f(x)\,dx$ is a continuous fun & is increasing.

Thus, $b_n := F(n)$ is an increasing sequence, and

$$b_n = F(n) \le F(t) \le F(n+1) = b_{n+1}, \quad \forall\ n \le t \le n+1.$$

If $\int_1^\infty f(x)\,dx$ conv. then by def of improper integrals, it means $\lim_{t\to\infty} F(t)$ exist $\Rightarrow$ $F(t)$ is bounded above
($\because$ $F(t)$ is increasing)

$\Rightarrow$ $\{b_n := F(n)\}$ is bounded above $\Rightarrow$ $\{b_n\}$ conv
(monotonic seq. thm)

O.T.H. if $\{b_n\}_{n=1}^\infty$ conv $\Rightarrow$ $\{b_n\}$ is bounded

i.e. $\exists\ M > 0$ s.t. $|b_n| \le M$, $\forall n$

But $b_n := F(n)$, thus $F(t)$ is bounded above

$\Rightarrow$ $\lim_{t\to\infty} F(t)$ conv. (b/c $F(t)$ is increasing & cont.)

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ conv.

If $F(t)$ is a cont fun. on $\mathbb{R}$, and is increasing.
If $F(t)$ is bounded above, then $\lim_{t\to\infty} F(t)$ exists.

means, if $\lim_{t\to\infty} F(t) = L$, then $\forall \varepsilon > 0$, $\exists\ x_0$, s.t. for $t > x_0$, $|F(t) - L| < \varepsilon$. In this case here, L is the least upper bound of the set of value of $F(t)$.

④ exercise

⑤ since $\frac{a_k}{b_k} \to L > 0$, choose ε small so that $L-\varepsilon>0$

then $\exists\, k_0$ s.t. for $n>k_0$, $\left|\frac{a_k}{b_k} - L\right| < \varepsilon$

i.e. $0 < L-\varepsilon < \frac{a_k}{b_k} < L+\varepsilon$

$\Rightarrow 0 < (L-\varepsilon) b_k < a_k < (L+\varepsilon) b_k$ $\quad$ ($\because a_k > 0,\ b_k > 0$)

Thus, if Σa_k conv. then

$(L-\varepsilon) b_k < a_k \Rightarrow \Sigma (L-\varepsilon) b_k$ conv. $\Rightarrow \Sigma b_k$ conv.

if Σb_k conv. then $\Sigma (L+\varepsilon) b_k$ conv.

so $a_k < (L+\varepsilon) b_k \Rightarrow \Sigma a_k$ conv. #

⑥ Ratio test : $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$ (idea : geometric series)

(i) $\lambda < 1$. So pick μ s.t. $\lambda < \mu < 1$. Then since

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$, there $\exists\, k_0$, s.t. for $k \geq k_0$,

(但綠色不會用到) $\lambda - (\mu - \lambda) < \frac{a_{k+1}}{a_k} < \mu = \lambda + (\mu - \lambda)$

$\Rightarrow a_{k_0+1} < a_{k_0} \cdot \mu \Rightarrow a_{k_0+2} < a_{k_0+1} \cdot \mu < a_{k_0} \cdot \mu^2$

$\Rightarrow a_k < a_{k_0} \cdot \mu^{k-k_0}$, $\forall\, k > k_0$

$\Rightarrow \sum_{k=k_0}^{\infty} a_k \leq \sum_{k=k_0}^{\infty} a_{k_0} \mu^{k-k_0} = \frac{a_{k_0}}{\mu^{k_0}} \boxed{\sum_{k=k_0}^{\infty} \mu^k}$ $\quad$ conv. b/c $|\mu|<1$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ conv.

(ii) if $\lambda > 1$, pick $1 < \mu < \lambda$, then $\exists\, k_0$, s.t.

for $k \geq k_0$, $\lambda-(\lambda-\mu) = \mu < \dfrac{a_{k+1}}{a_k} < \lambda + (\lambda-\mu)$

$\Rightarrow a_{k_0+1} > a_{k_0}\cdot\mu \Rightarrow a_k > a_{k_0}\cdot\mu^{k-k_0} \quad \forall\, k > k_0$

$\Rightarrow \sum_{k=k_0}^{\infty} a_k > \sum_{k=k_0}^{\infty} a_{k_0}\,\mu^{k-k_0} = \dfrac{a_{k_0}}{\mu^{k_0}} \boxed{\sum_{k=k_0}^{\infty} \mu^k}$ div b/c $|\mu| > 1$

(iii) if $\lambda = 1$, $\sum_{k=1}^{\infty} \frac{1}{k} \Rightarrow \frac{k}{k+1} \to 1$ while $\sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow \frac{k^2}{(k+1)^2} \to 1$

div conv.

so it's unknown.

Root test : $\lim_{k\to\infty} \sqrt[k]{a_k} = \rho$

(i) if $\rho < 1$, choose μ s.t. $\rho < \mu < 1$

Since $\lim_{k\to\infty} \sqrt[k]{a_k} = \rho$, $\exists\, k_0$, s.t. for $k > k_0$,

$\rho-(\mu-\rho) < \sqrt[k]{a_k} < \mu = \rho-(\mu-\rho) \Rightarrow a_k < \mu^k \quad (\because a_k > 0)$

for $k > k_0$

$\Rightarrow \sum_{k=k_0}^{\infty} a_k < \boxed{\sum_{k=k_0}^{\infty} \mu^k}$ conv. b/c $|\mu| < 1$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ conv.

(ii) if $\rho > 1$, choose μ s.t. $1 < \mu < \rho$

since $\lim_{k\to\infty} \sqrt[k]{a_k} = \rho$, $\therefore \exists\, k_0$ s.t. for $k > k_0$,

$\rho-(\rho-\mu) = \mu < \sqrt[k]{a_k} < \rho+(\rho-\mu) \Rightarrow \mu^k < a_k$ for $k > k_0$

$\Rightarrow \sum_{k=k_0}^{\infty} a_k > \boxed{\sum_{k=k_0}^{\infty} \mu^k}$ div b/c $|\mu| > 1$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ div.

(iii) $\rho = 1$. consider $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$, then both satisfy $\sqrt[k]{a_k} \longrightarrow 1$, but one series conv. while the other diverges.