Let S be a nonempty set of real numbers,

- a real number $\xi \in \mathbb{R}$ is called an upper bound of the set S, if

  $\xi \geq x, \ \forall x \in S$

  e.g. • let $S = \{1, 2, 3, 4\}$, then $4, 5, 6.8$, etc. are all upper bound of S

  • let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$, then $1, 2, 3.7, \pi$, etc, are all upper bound of S

- a real number $\eta \in \mathbb{R}$ is called a least upper bound of the set S, if

① η is an upper bound of S, and ② if ξ is any upper bound of S, $\xi \geq \eta$.

(i.e. η is the minimum among all upper bounds)

⑥ Least upper bound axiom

Every nonempty set of real numbers that has an upper bound has a least upper bound.

(clearly, if we change real numbers to rational numbers, it's not true)

Lemma A: Let f be a continuous fun on $[a,b]$. If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$ then $\exists\, c$, $a < c < b$, s.t. $f(c) = 0$

Pf: we'll prove the case for $f(a) < 0 < f(b)$, because $f(b) < 0 < f(a)$ is similar.

Now since $f(a) < 0$, f cont, there exists a $t > a$, s.t. $f(x) < 0\ \forall x \in [a,t)$

In fact, there are many ξ having this property.

Consider the set $\{t : f(x) < 0\ \forall x \in [a,t)\}$

Since this set has an upper bound, (for example, b is an upper bound.) it must have a least upper bound by the axiom.

Define $c :=$ least upper bound of S, where $S := \{t : f(x) < 0 \ \forall x \in [a, t)\}$. clearly $c \le b$.

":=" this notation means "defined as"

Moreover,

1° If $f(c) > 0$, then since f is cont., there exists $\eta > 0$, s.t. $\forall x \in (c-\eta, c]$, $f(x) > 0$. But this means that η is an upper bound of the set S and it is smaller than c, which contradict to the definition that c is l.u.b. $\times$ Thus $f(c) \le 0$.

2° Since $f(b) > 0 \Rightarrow c \ne b \Rightarrow c < b$.

3° If $f(c) < 0$, then since f is cont, there exists $\delta > 0$, s.t. $\forall x \in [c, c+\delta)$, $f(x) < 0$. But this means c is not an upper bound of the set S, also contradict to the definition that c is l.u.b. $\times$

Thus $f(c)=0$, and we are done #

Using this Lemma A, we can prove the intermediate value theorem:

Thm: If f is cont. on $[a,b]$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c)=k$.

Pf: suppose we have $f(a)<k<f(b)$. The other cases can be proved similarly.

Consider a new function $g(x):=f(x)-k$, then $g(a)<0$ and $g(b)>0$. So Lemma A implies that $\exists\, c$ between a and b s.t. $g(c)=0$, which means $f(c)=k$

#

Next, we look at the extreme value theorem.

Lemma B: If f is continuous on $[a,b]$, then f is bounded on $[a,b]$.

Pf. The idea is similar to Lemma A we'll consider a corresponding set, and argue that the least upper bound is what we want.

Consider a set S

$$S := \{t : t \in [a,b] \text{ and } f \text{ is bounded on } [a,t]\}$$

This set is nonempty because $a \in S$, it's bounded above by b because $S \subseteq [a,b]$

"subset"

Define $c :=$ least upper bound of S.

Clearly $c \le b$. Claim: $c = b$

Suppose $c < b$. Since f is continuous on $[a,b]$, it is cont. at c, so $\exists\, \eta > 0$ s.t. for $x \in [c-\eta, c+\eta]$, $|f(x) - f(c)| < 1$

i.e. $f(x)$ is bounded on $[c-\eta, c+\eta]$.

Since c is the l.u.b. of $S \Rightarrow c-\eta \in S$

So $f(x)$ is bounded on $[a, c-\eta]$.

But this means $f(x)$ is actually bounded on $[a, c+\eta]$, i.e. $c+\eta \in S$, contradict to the definition that c is l.u.b. of S.

$\therefore$ $c = b$.

This also means $f(x)$ is bounded on $[a, t]$ for all $t < b$, b/c now b is the l.u.b. of S.

On the other hand, f being continuous on $[a,b]$ implies that $\exists\, \delta > 0$, s.t. for $x \in [b-\delta, b]$, $|f(x) - f(b)| < 1$, i.e. $f(x)$ is bounded on $[b-\delta, b]$.

Now b being the l.u.b. of S implies that f is bounded on $[a, b-\delta]$, thus,

f is bounded on $[a, b]$.

\#

we use one more property : (Weierstrass Principal)

Every bounded infinite sequence of real numbers has a convergent subsequence.

(This property can be proved by using l.u.b.)

Thm : If f is continuous on a bounded closed interval $[a,b]$, then f takes on both a max value M and a min value m on $[a,b]$.

Pf: Since f is cont. on $[a,b]$, so by Lemma B, f is bounded on $[a,b]$, i.e. the set of value of f, $S := \{ f(x), x \in [a,b] \}$ is a bounded set. Then, by the l.u.b. axiom, there exists a l.u.b. M of S, i.e. M is the smallest number that satisfies $f(x) \le M, \forall x \in [a,b]$

$\Rightarrow$ either ① $M \in S$, then we are done

③ $M = \lim_{n\to\infty} a_n$, for $\{a_n\} \subset S$

(we'll look at sequences & their limits in the Spring)

In case ②, there exists

a sequence $\{x_n\} \subset [a,b]$ s.t.

$\lim_{n\to\infty} f(x_n) = M$ $\left(\begin{array}{l} \text{notice } \{x_n\} \subset [a,b] \\ \text{while } \{a_n\} \subset S \end{array}\right)$

domain of f

image of f

Then by the above property, there exists a convergent subsequence $\{y_n\} \subset \{x_n\}$, i.e.

$\lim_{n\to\infty} y_n = c$, for some $c \in [a,b]$.

Now, use again that f is continuous, we have

$$M = \lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty} y_n) = f(c).$$

minimum can be proved similarly.

#

Remark: Extreme value theorem can be proved by different methods. Here is just one of them.