Q: If we know that $\lim_{x\to\infty} f(x) = \infty$, how do we know if there's an oblique asymptote?

There are two suggestions by some of you, let me try to state what they are:

⟨Suggestion I⟩

Calculate $\lim_{x\to\infty} f'(x)$, and if its a fixed number, called it "a", then next calculate $\lim_{x\to\infty}(f(x) - ax)$ called it "b", and the oblique asymptote is $y = ax + b$.

The idea is that, if an oblique asymptote exists, i.e. $f(x) = (ax+b) + R(x)$ for some $a, b \in \mathbb{R}$, and $\lim_{x\to\infty} R(x) = 0$.

so $f'(x) = a + R'(x)$

And one may guess that if $\lim_{x\to\infty} R(x) = 0$,

then $\lim_{x\to\infty} R'(x) = 0$ also, thus

$\lim_{x\to\infty} f'(x) = a$, and $f(x) - ax = b + R(x)$

$\therefore \lim_{x\to\infty} f(x) - ax = \lim_{x\to\infty} b + R(x) = b$

While the idea to get "b" is correct, the idea to get "a" has some problem.

The reason being

$$\boxed{\lim_{x\to\infty} R(x) = 0 \text{ doesn't imply } \lim_{x\to\infty} R'(x) = 0}$$

for example, let $R(x) = \frac{1}{x}\sin(x^2)$

Then $\lim_{x\to\infty} R(x) = 0$. But the limit of

$$R'(x) = \underbrace{2\cos(x^2)}_{\text{no limit}} - \underbrace{\frac{1}{x^2}\sin(x^2)}_{0}$$ doesn't exist!

as $x \to \infty$

To be more precise, let $f(x) = x + \frac{1}{x}\sin(x^2)$

then $\lim_{x\to\infty} f(x) = \infty$, so no horizontal asymptote.

$\lim_{x\to\infty} \frac{1}{x}\sin(x^2) = 0$, so $y = x$ is an oblique asy.

However, if you just take $f'(x)$, then you would have

$$\lim_{x\to\infty} f'(x) = 1 + 2\cos(x^2) - \frac{1}{x^2}\sin(x^2)$$

which doesn't exist, and you would be concluding that $f(x)$ has no oblique asy. b/c $\lim_{x\to\infty} f'(x)$ doesn't exist; which is incorrect. Thus, we see that using $\lim_{x\to\infty} f'(x)$ doesn't necessarily give us what we want, b/c $\lim_{x\to\infty} R(x) = 0$ doesn't imply $\lim_{x\to\infty} R'(x) = 0$

< Suggestion II >

Calculate $\lim_{x\to\infty} \frac{f(x)}{x}$ and if its a fixed number, called it "a", then next calculate $\lim_{x\to\infty}(f(x) - ax)$ called it "b", and the oblique asymptote is $y = ax + b$.

The idea for this approach, is that, if the oblique asymptote exists, then

$f(x) = ax + b + R(x)$ with $\lim_{x\to\infty} R(x) = 0$

thus $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{ax}{x} + \frac{b}{x} + \frac{R(x)}{x}$

and since $\lim_{x\to\infty} \frac{b}{x} = 0$, $\lim_{x\to\infty} \frac{R(x)}{x} = 0$

we have $\lim_{x\to\infty} \frac{f(x)}{x} = a$, and

$\lim_{x\to\infty} f(x) - ax = \lim_{x\to\infty} b + R(x) = b$.

This approach is indeed correct, because we are only using what we know, namely,

$$\lim_{x\to\infty} R(x) = 0 \Rightarrow \lim_{x\to\infty} \frac{R(x)}{x} = 0$$

The meaning of this $\boxed{\lim_{x\to\infty} \frac{f(x)}{x}}$ limit is comparing the value of f(x) and the value of x, as $x \to \infty$, Thus if it has an oblique asymptote, $\lim_{x\to\infty} \frac{f(x)}{x}$ should be a constant!

so let's test the example $f(x) = x + \frac{\sin(x^2)}{x}$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} 1 + \frac{\sin(x^2)}{x^2} = 1, \because \lim_{x\to\infty} \frac{\sin(x^2)}{x^2} = 0$$

so $a = 1$, and $b = \lim_{x\to\infty} f(x) - 1 \cdot x = \lim_{x\to\infty} \frac{\sin(x^2)}{x} = 0$

$\therefore$ the oblique asymptote is $y = x$,

which is correct!