open, closed, or whatever

Def: X is a top space. A collection $\mathcal{A}$ of subsets of X is said to be locally finite in X, if every pt of X has a nbd that intersects only finitely many elements of $\mathcal{A}$.

e.g. $X = \mathbb{R}$ with standard top.

$\mathcal{A} := \{(n, n+2) \mid n \in \mathbb{Z}\}$ is locally finite.

What about $\mathcal{A} := \{(-n, n) \mid n \in \mathbb{Z}\}$?

Lemma 39.1

Let $\mathcal{A}$ be a locally finite collection of subsets of X, then

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\bar{A}\}_{A \in \mathcal{A}}$ of the closures of elements of $\mathcal{A}$ is locally finite.

(c) $\overline{\bigsqcup_{A \in \mathcal{A}} A} = \bigsqcup_{A \in \mathcal{A}} \bar{A}$ (compare with §17, ex6 (c))

pf: (a) trivial

(b) Any open set U that intersects $\bar{A}$ must also intersects A (why? b/c if $x \in \bar{A}$, then $U_x \cap A$ is nonempty $\therefore$ if Any $U \cap \bar{A} \neq \phi$, then U can be the nbd of some pt in $U \cap \bar{A} \subset \bar{A}$, then $U \cap A \neq \phi$) Thus if U is a nbd of x that intersects only finitely many elements A of $\mathcal{A}$, then U can intersect at most the same number of sets of the collection $\mathcal{B}$. (since it's possible that $\bar{A_1} = \bar{A_2}$ but $A_1 \neq A_2$)

(c) In general, we have $\bigcup_{A\in\mathcal{A}} \bar{A} \subset \overline{\bigcup_{A\in\mathcal{A}} A}$ (§17, ex6 (c))

To prove the other direction, let $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$.

let U be a nbd of x that intersects only finitely many elements of $\mathcal{A}$, say $A_1, \cdots, A_k$. If $x \notin$ one of $\bar{A_1}, \cdots, \bar{A_k}$, then $U \setminus \bar{A_1} \setminus \bar{A_2} \setminus \cdots \setminus \bar{A_k}$ will be a nbd of x that intersects no element of $\mathcal{A}$, and thus doesn't intersect their union $\bigcup_{A\in\mathcal{A}} A$ either, which contradicts to the assumption that $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$.

$\Rightarrow x$ must lie in one of $\bar{A_1}, \cdots, \bar{A_k} \Rightarrow x \in \bigcup_{A\in\mathcal{A}} \bar{A}$ ※

Def: A collection $\mathcal{B}$ of subsets of X is said to be countably locally finite if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

Def: Let $\mathcal{A}$ be a collection of subsets of the space X. A collection $\mathcal{B}$ of subsets of X is said to be a refinement of $\mathcal{A}$ if $\forall B \in \mathcal{B}$, there's an $A \in \mathcal{A}$ containing B. Such $\mathcal{B}$ is called an open refinement if all $B \in \mathcal{B}$ are open; is called a closed refinement if all $B \in \mathcal{B}$ are closed.

Lemma 39.2

Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{D}$ of X, refining $\mathcal{A}$ that is countably locally finite.

pf: Using well-ordering theorem: (cf: §10)

Def: a set A with an ordering is said to be well-ordered if every nonempty subset of A has a smallest element.

Thm: If A is a set, there exists an ordering on A that's well-ordering.

Choose a well-ordering "$<$" for $\mathcal{A}$ (open covering of X). Choose a metric d for X. Let $n \in \mathbb{Z}_+$.

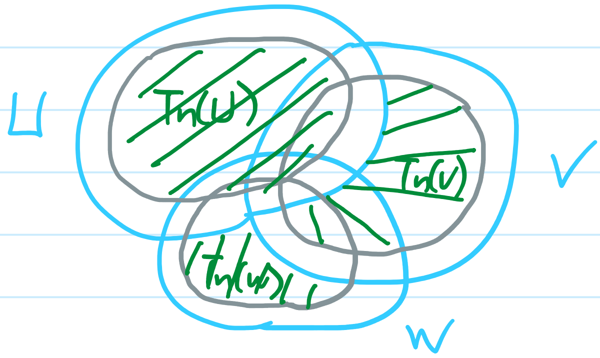
Given any $U \in \mathcal{A}$, define $S_n(U)$ by $\{x \mid B_d(x, \frac{1}{n}) \subset U\}$.

Then define $T_n(U) = S_n(U) \setminus \bigcup_{V<U} V$ (扣掉所有order更低的elements)

Then $T_n(U) \cap T_n(V) = \phi$ if $U \neq V$:

圖示: if $U < V < W$ (in ordering)

Then $T_n(U) = S_n(U)$, $T_n(V) = S_n(V) \setminus U$, $T_n(W) = S_n(W) \setminus U \setminus V$



$S_n(U)$, $S_n(V)$, $S_n(W)$ 內縮 $\frac{1}{n}$ distance.

$T_n(U)$, $T_n(V)$, $T_n(W)$ so clearly $T_n(U)$ etc disjoint.

now since $S_n(V) = \{x \mid B(x, \frac{1}{n}) \subset V\}$, we have, if $V \neq W \in \mathcal{A}$

then $d(x,y) \geq \frac{1}{n}$, $\forall x \in T_n(V)$, $y \in T_n(W)$

(if $x \in T_n(V)$, then $x \in S_n(V) \Rightarrow B(x, \frac{1}{n}) \subset V$, but $V < W \Rightarrow V \cap T_n(W) = \phi$

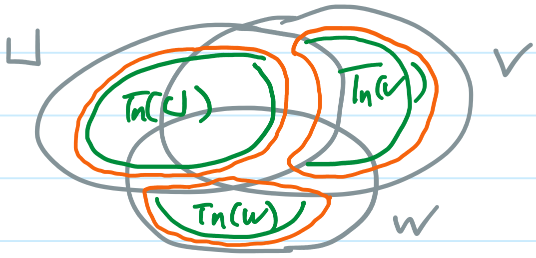
and $y \in T_n(W) \Rightarrow d(x,y) \geq \frac{1}{n}$)

so far, we have $T_n(U) \subseteq S_n(U) \subsetneq U$, but $T_n(U)$ is not open.

our goal is to find an open covering refining $\mathcal{A}$, so let's modify $T_n(U)$ further:

Define $E_n(U) = \bigcup_{x \in T_n(U)} B_d(x, \frac{1}{3n})$ $\therefore$ $T_n(U) \subset E_n(U) \subset U$

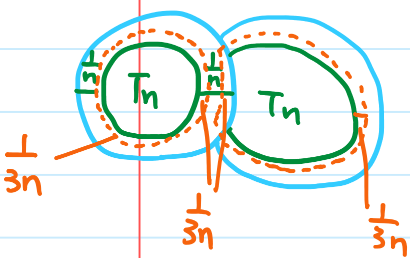
moreover, assuming the ordering is $U < V < W$, then we have the corresponding $E_n(U)$, $E_n(V)$, $E_n(W)$. They are all disjoint b/c if $x \in E_n(V)$, $y \in E_n(W)$, then $d(x,y) \geq \frac{1}{3n}$



Since $d(x,y) \geq \frac{1}{n}$ if $x \in T_n(V)$, $y \in T_n(W)$ and $E_n(V)$, $E_n(W)$ are "fattened" $T_n(V)$, $T_n(W)$ by $\frac{1}{3n}$

$\Rightarrow$ they still have a distance of $\frac{1}{3n}$ left.

放大來看：

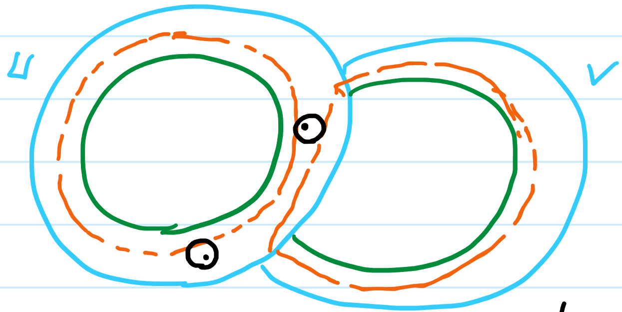


($\because E_n(U) \subset U$)

Define $\mathcal{O}_n = \{ E_n(U) \mid U \in \mathcal{A} \}$ is a refinement of $\mathcal{A}$, that is also locally finite.

$\mathcal{O}_n$ is locally finite, b/c,

$\forall x \in X$, $B_d(x, \frac{1}{6n})$ can intersect at most one element of $\mathcal{O}_n$



$B_d(x, \frac{1}{6n})$ has diameter $\frac{1}{3n}$

so clearly $\mathcal{O}_n$ doesn't cover X.

But we claim that $\mathcal{O} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{O}_n$ do (which is a countable refinement of $\mathcal{A}$)

($\because$ well-ordered)

$\forall x \in X$, since $\mathcal{A}$ covers X, we can find the smallest element U in $\mathcal{A}$ that contains X. Since U is open, $\exists\ B_d(x, \frac{1}{n}) \subset U$ for some n, i.e. $x \in S_n(U)$ for some n.

Now since U is the smallest element in $\mathcal{A}$ that contains x, $x \in T_n(U)$ also $\Rightarrow x \in E_n(U) \supset T_n(U) \Rightarrow x \in \mathcal{O}_n$ for some n

$\Rightarrow \mathcal{O} = \bigsqcup_{n \in \mathbb{Z}_+} \mathcal{O}_n$ covers X. #

Def: A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

An equivalent way of saying "compactness":

A space X is cpt if every open covering $\mathcal{A}$ of X has a finite open refinement $\mathcal{B}$ that covers X.

Why equivalent? Given $\mathcal{B}$ refinement of $\mathcal{A}$, $\forall B \in \mathcal{B}$, choose $A \in \mathcal{A}$ s.t. $B \subset A$. Since $\mathcal{B}$ is finite, it gives us a finite subcollection of $\mathcal{A}$ that covers X!

From this, we see compact imply paracompact but not vice versa:

Example: $\mathbb{R}$ is paracpt but not cpt:

We know $\mathbb{R}$ is not cpt. To show that $\mathbb{R}$ is paracompact, Let $\mathcal{A}$ be any open covering of $\mathbb{R}$. Let $B_0 = \phi$, $B_1 = (-1,1)$, ... $B_n = (-n,n)$, ... a countable collection of open sets.

Given $m \in \mathbb{Z}$, $[-m,m]$ is cpt, so we can find finitely many elements of $\mathcal{A}$ that covers $[-m,m]$. Denote them by $A_1^m, \cdots, A_{k_m}^m$. Intersect $A_1^m, \cdots, A_{k_m}^m$ with $\mathbb{R} \setminus [-m+1, m-1]$, and denote this finite collection by $\mathcal{C}_m$

所以每次都把上一回的扣掉!!

i.e. $\mathcal{C}_m = \{ A_i^m \cap (\mathbb{R} \setminus [-m+1, m-1]) \}$

Then $\mathcal{C} = \bigsqcup_m \mathcal{C}_m$ is a refinement of $\mathcal{A}$, that is locally finite and a covering of $\mathbb{R}$ $\Rightarrow$ $\mathbb{R}$ is paracpt.

There's some similarity between cpt & paracpt: Recall Thm 26.2 says, every closed subspace of a cpt space is cpt.

Thm 41.2 Every closed subspace of a paracompact space is paracompact.

pf: Let Y be a closed subspace of X, which is paracpt. let $\mathcal{A}$ be an open covering of Y, i.e. if $A \in \mathcal{A}$, then $A = A' \cap Y$ for some A' open in X. Denote this collection by $\mathcal{A}'$, then $\mathcal{A}' \cup (X \setminus Y)$ is an open covering for X.

Since X is paracpt, it has a locally finite open refinement $\mathcal{B}$ of this covering that covers X. Thus $\mathcal{C} = \{ B \cap Y \mid B \in \mathcal{B} \}$ is the desired refinement of $\mathcal{A}$.

\#

But there is some clear difference between cpt and paracpt:

e.g. Thm 26.3 says, every cpt subspace of a Hausdorff space is closed. But a paracpt subspace of a Hausdorff space may not be closed: Recall $\mathbb{R}$ is Hausdorff.

Consider $(0,1)$, it's homeomorphic to $\mathbb{R}$, thus it's paracpt, but clearly it's not closed in $\mathbb{R}$.

Lemma 41.3 (E. Michael)

Let X be regular. Then the following are equivalent:

Every open covering of X has a refinement that's

(1) An open covering of X and countably locally finite.

(2) A covering of X and locally finite.

(3) A closed covering of X and locally finite

(4) An open covering of X and locally finite.

trivial

Pf: (1) $\Rightarrow$ (2)

Assume the open covering $\mathcal{A}$ of X has a refinement $\mathcal{B}$ that's an open covering of X and is countably locally finite.
We want to find a refinement $\mathcal{C}$ of $\mathcal{A}$ that's locally finite and covers X

Since $\mathcal{B}$ is countably locally finite, we can write $\mathcal{B} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite.

For each $n \in \mathbb{Z}_+$ and $U \in \mathcal{B}_n$, define

$$S_n(U) = U \setminus \bigcup_{i<n} \left(\bigcup_{U\in\mathcal{B}_i} U \right)$$

← 所以 $S_n(U)$ 不一定 open, closed, whatever

idea: 把 U 中所有与 $\mathcal{B}_1, \cdots, \mathcal{B}_{n-1}$ 有重叠的都扣掉，所以 $S_n(U)$ 与 $\mathcal{B}_1, \cdots, \mathcal{B}_{n-1}$ 都没交集!

Let $\mathcal{C}_n = \{ S_n(U) \mid U \in \mathcal{B}_n \}$

Then $\mathcal{C}_n$ is a refinement of $\mathcal{B}_n$ ($\because S_n(U) \subset U$)

claim: $\mathcal{C} = \bigcup_n \mathcal{C}_n$ is a covering of X and is locally finite.

- If $x \in X$, let n_0 be the smallest integer that $x \in \mathcal{B}_{n_0}$. Let $U \in \mathcal{B}_{n_0}$ and U contains x. Since n_0 is the smallest integer, x doesn't belong to any $\mathcal{B}_i$ if $i < n_0$

By def of $S_{n_0}(U)$, $x \in S_{n_0}(U) \in \mathcal{C}_{n_0} \subset \mathcal{C}$ $\therefore$ $\mathcal{C}$ covers X.

- To show that $\mathcal{C}$ is locally finite, for $x \in X$, we need to find a nbd st. it intersect only finitely many elements of $\mathcal{C}$.

First, notice that $\mathcal{B}_n$ is locally finite $\forall n$ by assumption.

Let n_0 be the smallest integer that $x \in \mathcal{B}_{n_0}$, then $\forall n \leq n_0$, we can choose a nbd W_n of x s.t. it only intersects finitely many elements of $\mathcal{B}_n$ ($\because$ $\mathcal{B}_n$ is locally finite)

(雖然 x 不在 $\mathcal{B}_n$ 當 $n < n_0$, 但 x 的 nbd 可以有交集)

In fact, W_n only intersects finitely many elements of $\mathcal{C}_n$, because, if $W_n \cap \underbrace{S_n(V)}_{\in \mathcal{C}_n} \neq \phi$ for some $V \in \mathcal{B}_n$, then $W_n \cap V \neq \phi$ $\because S_n(V) \subset V$. But since W_n only intersects finitely many elements of $\mathcal{B}_n$, it can only intersects finitely many elements of $\mathcal{C}_n$.

Moreover, since $x \in U \in \mathcal{B}_{n_0}$ (we chose this), and for any n, $S_n(U) \cap \mathcal{B}_i = \phi$, $\forall i < n$, thus U can not have any intersection with $S_n(V)$, $\forall n > n_0$, $V \in \mathcal{B}_n$

$\Rightarrow$ $U \cap \mathcal{C}_n = \phi$ $\forall n > n_0$

$\Rightarrow$ $W_1 \cap W_2 \cap \cdots \cap W_{n_0} \cap U$ is a nbd of x that intersects only finitely many elements of $\mathcal{C}$

(2) ⇒ (3) "Every open covering of X has a refinement that is a covering of X and locally finite" ⇒ "Every open covering of X has a refinement that's a closed covering of X and locally finite".

Let $\mathcal{A}$ be an open covering of X. Since $\mathcal{A}$ is an open covering, for any $x \in X$, $x \in A$ for some $A \overset{\text{open}}{\in} \mathcal{A}$. Since X is regular, $\exists$ a nbd W_x of x s.t. $\overline{W_x} \subset A$. Let $\mathcal{B}$ be the collection of all open sets $U \subset X$ s.t. $\overline{U} \subset A$ for some $A \in \mathcal{A}$. Clearly $\mathcal{B}$ covers X.

Now we'll apply (2) to $\mathcal{B}$: By assumption (2), $\mathcal{B}$ has a refinement that is a covering of X and locally finite. Call this refinement $\mathcal{C}$. Let $\mathcal{D} = \{ \overline{V} \mid V \in \mathcal{C} \}$. Then $\mathcal{D}$ covers X b/c $\mathcal{C}$ covers X, it's locally finite by Lemma §41.1(b). It's a refinement of $\mathcal{A}$ b/c $\overline{V} \subset \overline{U}$ for some $U \in \mathcal{B}$ and $\overline{U} \subset A$ for some $A \in \mathcal{A}$. The reason that we don't use $\mathcal{A}$ directly to get the $\mathcal{C}$ then $\mathcal{D}$ is b/c if $\mathcal{C}$ is a refinement of $\mathcal{A}$ then $\overline{V} \subset \overline{A}$ not necessarily $\overline{V} \subset A$, for $V \in \mathcal{C}$, which means $\mathcal{D}$ may not be a refinement of $\mathcal{A}$.

(3) ⇒ (4) Let $\mathcal{A}$ be an open covering of X. By (3), we have a closed covering $\mathcal{B}$ of X which is a refinement of $\mathcal{A}$ and is locally finite. We want to expand $B \in \mathcal{B}$ slightly to an open set, such that it remains locally finite, and still a refinement of $\mathcal{A}$.

Step 1°: define the open set (unlike "shrinking", we'll "expand")

$\mathcal{B}$ is locally finite, $\therefore \forall x \in X, \exists U_x$ s.t. U_x intersects only finitely many elements of $\mathcal{B}$. Then $\{U_x : x \in X\}$ is a covering of X, and by (3) again, it has a closed refinement $\mathcal{C}$ that is locally finite. And since it's a refinement of $\{U_x\}$, Any $C \in \mathcal{C}$ is a subset of some U_x which intersects finitely many elements of $\mathcal{B}$, so C can intersect at most that many element of $\mathcal{B}$ i.e. any $C \in \mathcal{C}$ intersects finitely many elements from $\mathcal{B}$.

Now for each $B \in \mathcal{B}$, define $\mathcal{C}(B) = \{C \mid C \in \mathcal{C}, C \cap B = \phi\}$

i.e. 那些不跟B相交的C

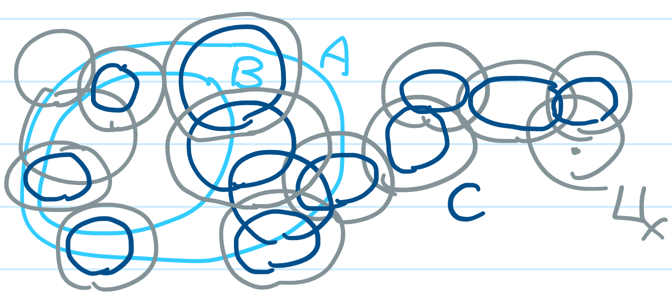
Then define $E(B) = X \setminus \bigcup_{C \in \mathcal{C}(B)} C$, then $E(B) \supset B$

But $\mathcal{C}(B)$ is a subcollection of $\mathcal{C}$, so by Lemma 39.1 (a) $\Rightarrow \mathcal{C}(B)$ is also locally finite, (c) $\Rightarrow \overline{\bigcup_{C \in \mathcal{C}(B)} C} = \bigcup_{C \in \mathcal{C}(B)} \overline{C} \overset{\because C \text{ closed}}{=} \bigcup_{C \in \mathcal{C}(B)} C$

i.e. $\bigcup_{C \in \mathcal{C}(B)} C$ is closed $\Rightarrow E(B)$ is open. However, $E(B)$ may be too big and fail to be a refinement of $\mathcal{A}$. To correct this, $\forall B \in \mathcal{B}$, choose $F(B) \in \mathcal{A}$ containing B ($\because \mathcal{B}$ is a refinement of $\mathcal{A}$). Define $\mathcal{D} = \{E(B) \cap \underbrace{F(B)}_{\in \mathcal{A}} \mid B \in \mathcal{B}\}$

then this is a refinement of $\mathcal{A}$,

and since $B \subset E(B) \cap F(B)\ \forall B$, $\mathcal{D}$ also covers X.



- $B \subset A$ refinement
- $C \subset U_x$ refinement

Step 2° show that $\mathscr{D}$ is locally finite.

Since $\mathscr{C}$ is locally finite, $\forall x \in X$, $\exists V_x$ s.t. V_x only intersect finitely many elements from $\mathscr{C}$, denote them by $C_1, \cdots, C_k$

Since $\mathscr{C}$ is a covering of X, we know

$$V_x = V_x \cap X = V_x \cap \left(\bigcup_{C \in \mathscr{C}} C \right) = V_x \cap \left(\bigcup_{i=1}^{k} C_i \right) \Rightarrow V_x \subset \bigcup_{i=1}^{k} C_i$$

claim: each C_i only intersects finitely many elements of $\mathscr{D}$.

If C_i has nonempty intersection with some element from $\mathscr{D}$, say $C_i \cap (E(B_0) \cap F(B_0)) \neq \phi$ for some $B_0 \in \mathscr{B}$.

Then $C_i \cap E(B_0) \neq \phi$, and since $E(B_0) := X \setminus \bigcup_{C \in \mathscr{C}(B_0)} C$

所有不跟 B_0 相交的 C

$\Rightarrow C_i \cap B_0 \neq \phi$

i.e. if C_i intersects $\overbrace{E(B_0) \cap F(B_0)}^{\in \mathscr{D}}$, then it must also intersects B_0. However, we know that C_i can only intersects finitely many elements of $\mathscr{B}$ (this is from our construction of the C)

$\Rightarrow$ C_i can intersects at most the same number of elements of $\mathscr{D}$

$\Rightarrow$ V_x intersects finitely many elements of $\mathscr{D}$, $\forall x \in X$

$\Rightarrow$ $\mathscr{D}$ is locally finite.

#