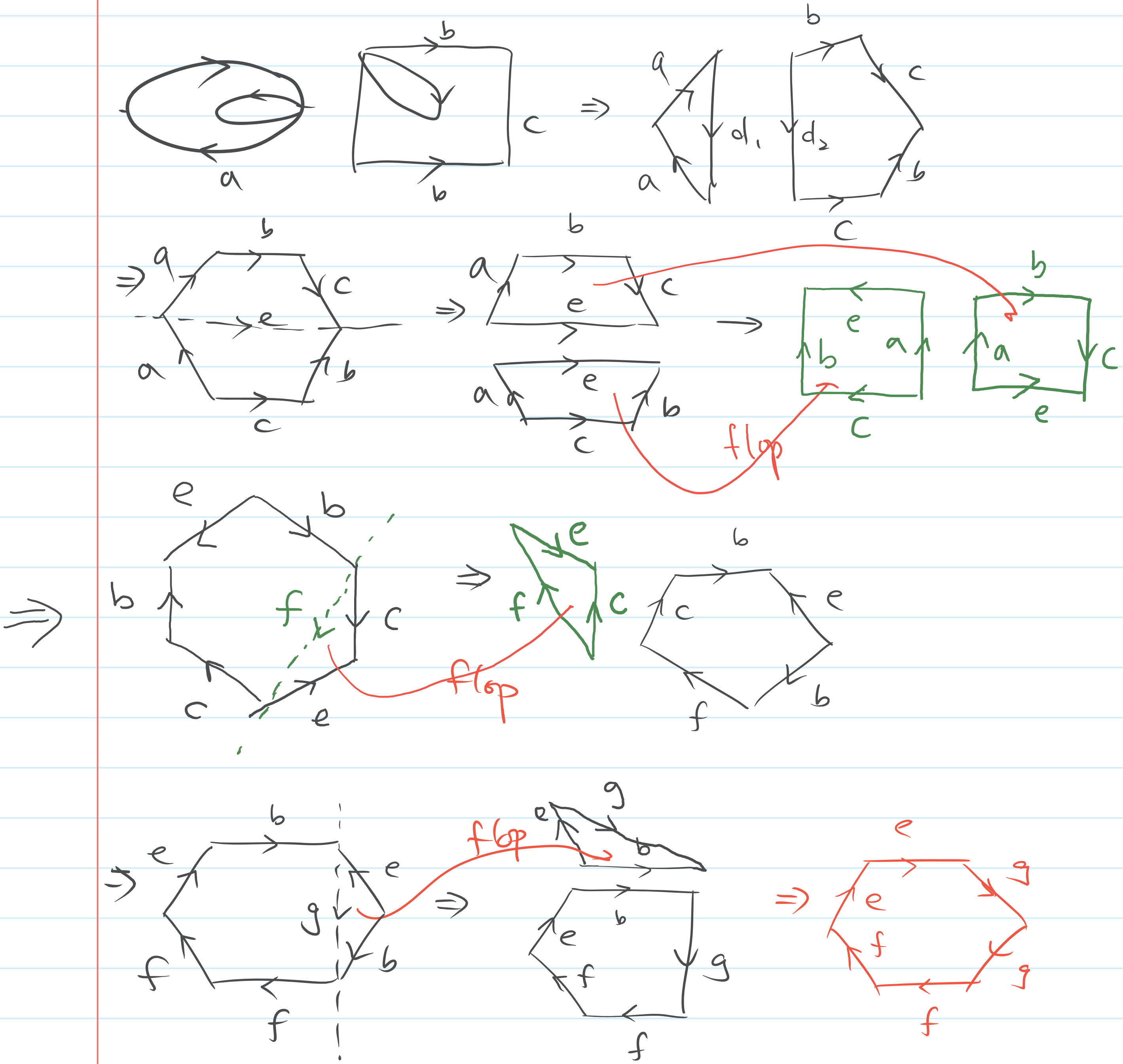
Lemma: $T \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$



$\underline{e^2 g^2 f^2}$

connected sum of 3 $\mathbb{RP}^2$

Thm: Any cpt surface is either homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes.

Examples: we know

Torus : $aba^{-1}b^{-1}$

$\mathbb{RP}^2$: aa

sphere : aa^{-1}

$\#_n T^2$: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$

$\#_n \mathbb{RP}^2$: $a_1a_1a_2a_2a_3a_3 \cdots a_na_n$

so this theorem claim that for any cpt surface, we can always show that it's one of the above.

the expression of the surface, or the "word" of the surface

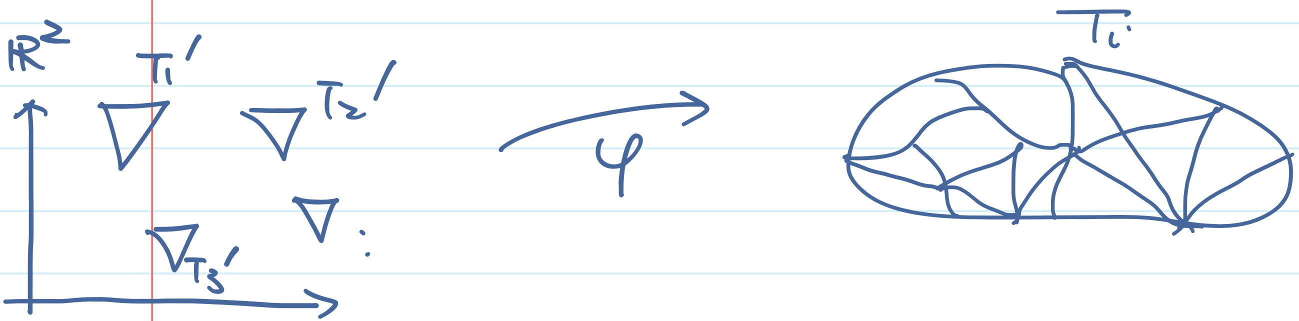
Step 1° Show that the surface is obtained from D (disc) by identifying certain paired edges on the boundary ∂D.

Striangulate S into $\bigsqcup_{i=1}^{n} T_i$ s.t. the triangle T_i has an edge e_i in common with at least one of the triangle T_j for some $1 \le j < i$.

This is possible, because we can number T_i in the following way call any Δ, T_1, to choose T_2, choose any Δ that has a common edge with T_1, and call the common edge e_2. To choose T_3, choose any triangle that has an edge in common with T_1 or T_2, and call this common edge e_3. Continue this process, we are done.

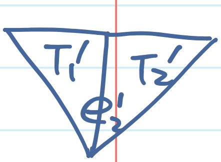
If at some point we can't continue, then it means we have at least two sets of triangles $I_1 = \{T_1, \cdots, T_k\}$, and $I_2 = \{T_{k+1}, \cdots, T_n\}$ s.t. any $T_i \in I_1$ has no edge or vertex in commmn with any $T_j \in I_2$ $\Rightarrow$ S is a union of two nonempty disjoint closed set $\rightarrow\!\!\!\!\!\!\times$ S is connected!

Then S can be obtained from D (disc) by identifying certain paired edges on ∂D by the following reason:

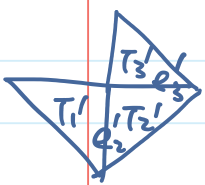


重排 $\downarrow$ (gluing or quotient)

polygon is obtained as a quotient space of T'



glue T_1' with T_2' (i.e. quotien e_2')



glue T_3' with T_2' (i.e. quotient e_3')

$\vdots$

i.e. glue 在一起

let D denote the resulting quotient space of T'

Then φ:

$$\begin{tikzcd} T' \arrow[r] \arrow[d, "/\sim"'] & S \\ T'/\sim := D \arrow[ur, "\psi"'] & \end{tikzcd}$$

induces a map ψ onto S and S has the quotient topology induced by ψ

The reason that $D := T'/\sim$ is topologically a disc is the following:

T_1' & T_2' are both top. equiv. to disc, so the quotient space of $T_1' \cup T_2' / \varphi^{-1}(e_2)$

called it D_2, is again a top. disc.

Now we form the next quotient space D_3 by $T_3' \sqcup D_2 / \varphi^{-1}(e_3)$

so this again is a top. disc.

Continue this process, we get $D = D_n = T_n' \sqcup D_{n-1} / \varphi^{-1}(e_n)$

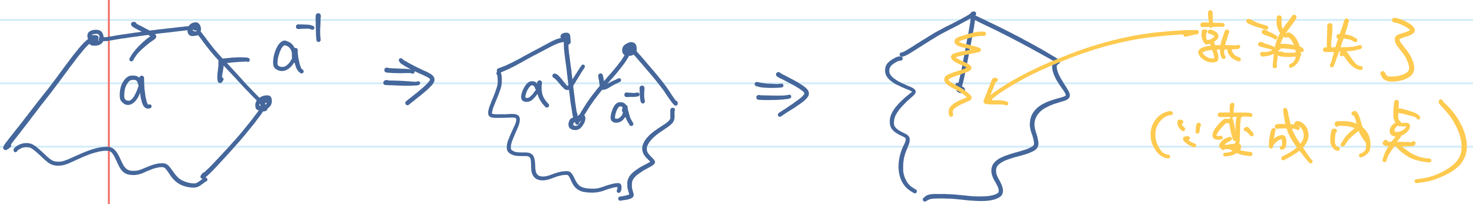
is a top. disc.

clearly S is obtained from D by identifying certain paired edges on ∂D, b/c S is cpt, i.e. without bdy, i.e. each edge is an interior edge, so they must appear as pair with opposite orientation.

Step 2°: Eliminate adjacent edge of first kind
相連的

first kind: aa^{-1} or $a^{-1}a$
second kind: aa or $a^{-1}a^{-1}$

If there are at least 4 edges in total (in the expression) we can eliminate the first kind:



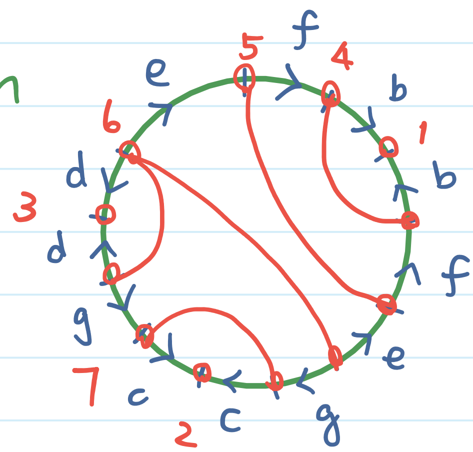
so we can eliminate all the adjacent first kind, unless ① it's a first kind with only 2 edges: aa^{-1}, then we can't continue, and this is exactly S^2. ② or it's $aa^{-1}bb$, and by eliminating aa^{-1}, we end up with one second type bb, and this is exactly $\mathbb{RP}^2$.

So now we are left with second kind of more than one pair, or mixed with first kind but not adjacent.

Step 3: transforming to a polygon s.t. all vertices must be identified to a single vertex:

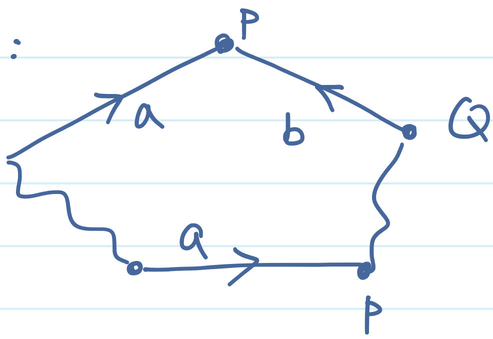
Introducing the idea of "equivalent vertices"

Recall the expression we saw earlier:



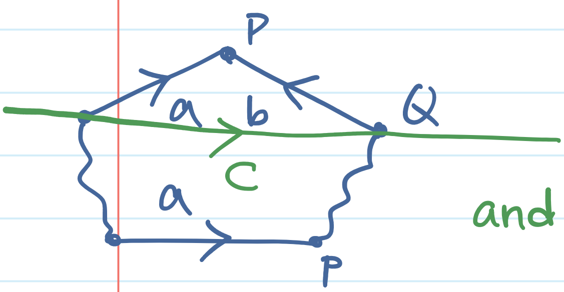
total 7 equivalent classes

So now suppose there are at least two different classes then at least we have an adjacent pair of vertices which are not equivalent:



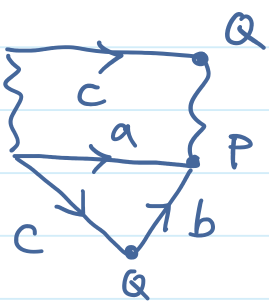
by step 2 we have eliminat aa^{-1} ∴ $b \neq a$, this is also not aa b/c we said we have two different classes of vertex so $P \neq Q$ and if this is aa then Q是P!

now cut thru



and glue along a

⇒



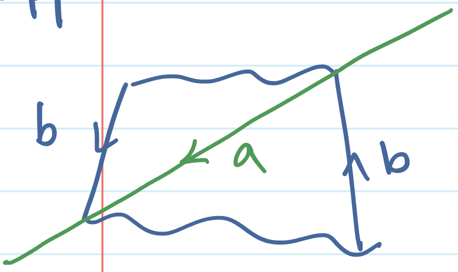
∴ #{P}−1, while #{Q}+1

(i.e. number of vertex of equiv. class p)

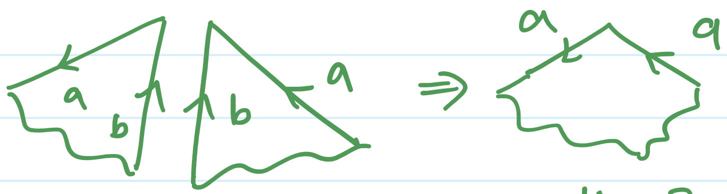
continue this process, (along the way, we may need to perform step 2 again b/c edges move in step 3), eventually {p} becomes an empty set, and continue this we get only one eq. class of vertex.

Step 4: make any pair of edges of the second kind adjacent.

Suppose not:



b b not adjacent, then cut along a and glue b, we get



now adj. 2nd kind.

Continue this, until all pairs of edges of the 2nd kind are adjacent. $\boxed{\cong \#_n \mathbb{RP}^2}$

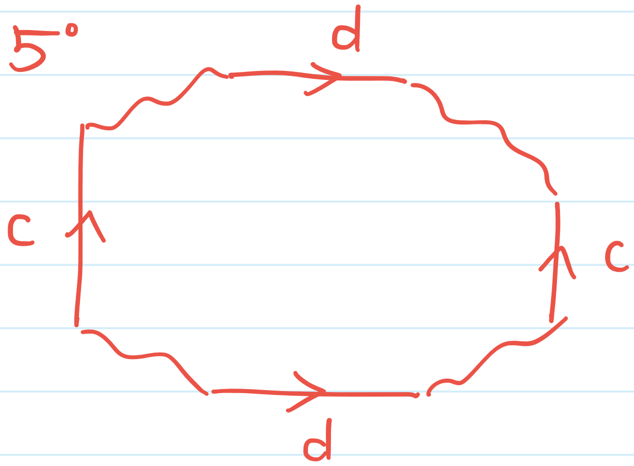
If there is no first kind in the expression, then we are done, and the surface is $a_1a_1a_2a_2\cdots a_na_n$

If there is first kind existing somewhere, say c, then $\exists$ at least one other pair of edges of first kind s.t. these two pairs seperate one another, i.e. $\cdots c \cdots d \cdots c^{-1} \cdots d^{-1} \cdots$

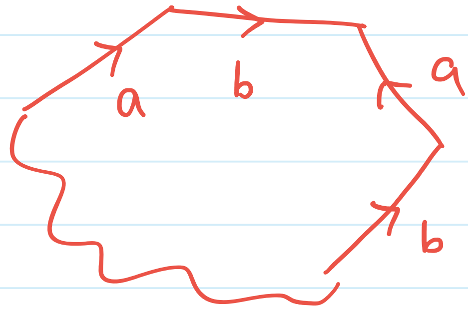
The reason is the following, if not, then

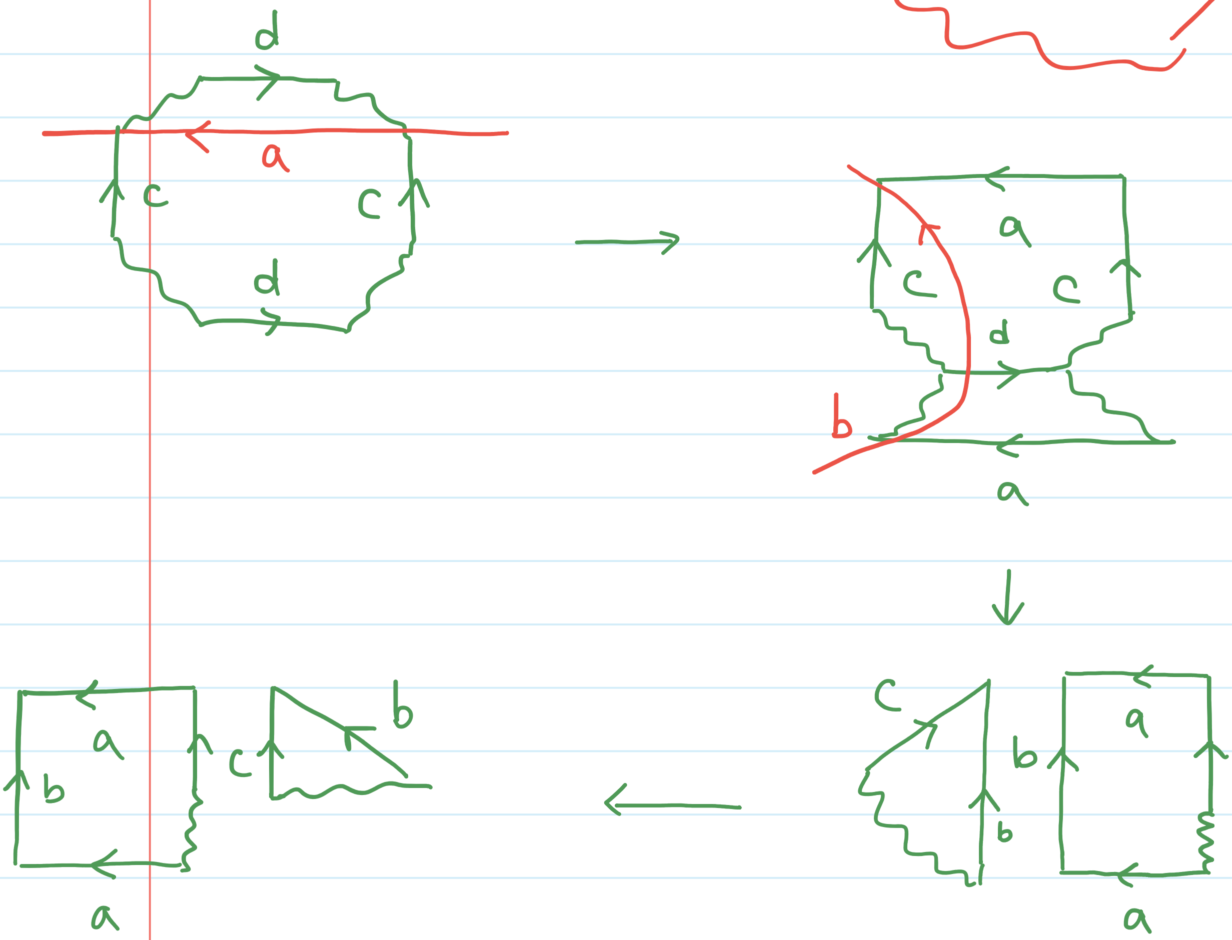
and all edges in A identify within A, all edges in B identify within B, then P and Q have no chane to be equivalent, but this contradict to step 3 that we have only one equiv. vertex.

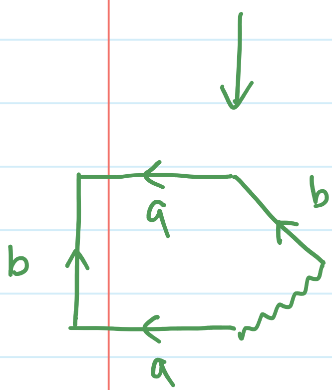
Step 5°



can be transformed into







done.

⇒ all pairs of the first kind are in adjacent gps of four as $aba^{-1}b^{-1}$.

If there are no pairs of the second kind, then we are done and the result is $\underset{n}{\#} T^2$.

If there are pairs of the first kind and second kind, then by Lemma that $T^2 \# \mathbb{RP}^2 \cong \underset{3}{\#} \mathbb{RP}^2$, we can obtain that the result is $\underset{m}{\#} \mathbb{RP}^2$. $\boxed{\text{QED}}$

Thm: S_1, S_2 are cpt, connected surfaces. Then S_1 is top. equiv. to S_2 iff $\chi(S_1) = \chi(S_2)$ and either both are orientable or nonorientable.

Exercise: $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$

$\chi(S^2) = 2$, $\chi(\underset{n}{\#} T^2) = 2 - 2n$, $\chi(\# \mathbb{RP}^2) = 2 - m$

for $S_1 \setminus D$: $E-3$, $F-1$, $V-3$

to D 是

$V=2$
$F=2$ $v=4$

Ref: Massey: A basic course in A.T.
L. C. Kinsey: Topology of Surface
J. Munkres: Elements of A.T.